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Symmetries and Invariant Solutions for the Thermal Expulsion Equation

Maha Falih Jassim
Department of Systems of Computers, Commission of Technical Education،
Technical Institute \ Kirkuk,Kurdistan Region Iraq.

Abstract
In this paper we study the classical Lie symmetries method for a two-dimensional partial differential equation (PDE) which is called the Thermal Expulsion Equation, and we obtained reductions to an ordinary differential equation of a second order (ODE) called principal (ODE). Then we analyzed some problems of the Thermal Expulsion Equation when it is invariant to the stretching group to derive an approximate solution of the Expulsion Equation corresponding to impulsive boundary conditions, when these conditions are clamped and a slowly varying happens in them with respect to time again.

Keyword: Lie symmetries, classical symmetries, similarity solutions, and invariant solutions

Introduction
Symmetry analysis plays an important rule in the theory of Differential Equation [1-3]. Finding a general solution for most of the Partial Differential Equations (PDEs) of science and engineering is too difficult. Instead, we are usually satisfied to find particular solutions defined by specific initial and boundary conditions. If the PDE is invariant to a group of transformation, then among its solutions will be some of their own images under transformation. These invariant solutions are generally easier to be calculate than the others; oftenly they may be calculated by solving an Ordinary Differential Equation (ODE). Original symmetry method for reducing the order of (ODEs) and the number of independent variables for both linear and non-linear(PDE). Probably the most useful point transformation of the (PDEs) is those, which form a continuous Lie point symmetry group (the classical Lie point symmetry method). The method for determining the symmetry group of differential equation is straightforward and described in several books as in [1- 5] and many papers like [3,6-15] Let

$F(z, t, T(z,t), T_z, T_t, T_{zt}, T_{tt}, T_{zz}) = 0, \quad a < z < b, \quad t > 0 \quad (1)$

be partial differential equation in one dependent variable T and two independent variables z and t ,and let $T = \Theta(z,t)$ be solution of eq.(1) ,then

$$T^* = T^*(T,z,t;\varepsilon)$$
$$z^* = z^*(T,z,t;\varepsilon) \qquad (2)$$
$$t^* = t^*(T,z,t;\varepsilon)$$

be the one parameter (ε) transformation group to the variables T, z and t , then the infinitesimal transformation to this group is

$$z^* = z + \varepsilon\, \xi(z,t,T) + O(\varepsilon^r) \quad (3)$$
$$t^* = t + \varepsilon\, \tau(z,t,T) + O(\varepsilon^r)$$
$$T^* = T + \varepsilon\, \eta(z,t,T) + O(\varepsilon^2)$$

If the eq.(1) is invariant under eq .(2), the infinitesimal transformation of eq.(1) is

$$V = \xi \partial z + \tau \partial t + \eta \partial T \qquad (4)$$

Where ξ, τ, η are functions of z,t,T. These coefficients can be determined from the invariant condition $\xi\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial t} + = \eta$ (5)

E-mail: maha_mm72@yahoo.com

the general solution to eq.(5) can be found by solving the characteristic equations

$$\frac{dz}{\xi}=\frac{dt}{\tau}=\frac{dT}{\eta} \qquad (6)$$

the general solution contains two constant, the first independent variable $x(z,t)$ is called similarity variable and the second is dependent variable $f(x)$, i.e. the similarity solution is $\Theta(z,t) = F(z,t,f(x))$

And by substituting this similarity solution in eq.(1), then the original PDE is reduced to ODE by variables x and $f(x)$.

If eq. (1) invariance under a one-parameter stretching group in [4-6]

$$T^*=\lambda^{\alpha}T \ , \ t^*=\lambda^{\beta}t \ , \ z^*=\lambda z, \qquad (7)$$

where λ is the group parameter that labels the individual transformation , α and β are parameters connected by a linear relation

$$M\alpha + N\beta = L \qquad (8)$$

where M, N, L are fixed constants determined by the structure of a particular PDE. Thus only one of the family parameters α and β can be chosen independently, as we see the values of α and β in particular problems are determined by the boundary and initial conditions. Then the characteristic equations are

$$\frac{dT}{\alpha T}=\frac{dz}{z}=\frac{dt}{\beta t} \qquad (9)$$

And by solving three independent integrals of eq. (9) we obtain

$$x = z / t^{1/\beta} \ , \ y(x) = T/t^{\alpha/\beta}$$

$$T(z,t) = t^{\alpha/\beta}\, y(x) \ , \quad x = z / t^{1/\beta} \qquad (10)$$

If eq.(10) is substituted into eq.(1) , an ordinary differential equation for y must result because y is function of only one argument namel $x = z / t^{1/\beta}$,the convenient name for this ODE is the principal differential equation because similarity solution can be found by solving the related ODE rather than PDE it self .

Now in this paper we are discussing the classical Lie point symmetries of a two-dimensional Thermal Expulsion equation,

$$T\,T_t = T_{zz} \qquad (11)$$

which arises in the theory of expulsion of compressible fluid from a long tube where T is the temperature, z one-dimension and t is the time in the semi – infinite pipe [5]; Also we are using the invariant of this Expulsion Thermal equation under the one - parameter stretching group which provides reduction to an ordinary differential equation of the a second order, then we analyzed some problems of the Expulsion Equation using similarity solution, when the boundary conditions are clamped and when they have a slowly varying to derive an approximate solution of the Expulsion equation corresponding to impulsive boundary condition.

2- The Thermal Expulsion Equation

2.1- Classical Point Symmetries of Expulsion Equation

We consider the one parameter (ε) Lie point transformation of (z,t,T) in given in eq.(3) then

$$T^*_{z^*} = T_z + \varepsilon\,[\eta_z] \ , \quad T^*_{t^*} = T_t + \varepsilon\,[\eta_t] \ ,$$

$$T^*_{z^*z^*} = T_{zz} + \varepsilon\,[\eta_{zz}] \qquad (12)$$

Its straightforward to establish from eqs.(10) the following relations in [8],[13]

$$[\eta_t]=\eta_t+(\eta_T-\tau_t)T_t-\xi_t\,T_z-\tau_T\,T_t^2-\xi_T\,T_z T_t \qquad (13)$$

$$[\eta_z]=\eta_z+(\eta_T-\xi_z)T_z-\tau_z T_t-\xi_T\,T_z^2-\tau_T\,T_z T_t \qquad (14)$$

$[\eta_{zz}]=$

$\eta_{zz} + (2\eta_{zT} - \xi_{zz})T_z - \tau_{zz}T_t + (\eta_T - 2\xi_Z)$

$T_{ZZ} - \tau_z T_{tz} + (\eta_{TT} - 2\xi_{zT})T_z^2 -$

$\xi_{TT}T_Z^3 - \tau_{tt}T_t T_x^2 - 3\xi_T T_Z T_{ZZ} - \tau_T T_t T_{zz} - 2\tau_T T_z T_{tz}$ (15)

The eq. (11) is invariant to the infinitesimal transformation in eq. (3) provided that the following conditions are satisfied:

$V = \xi\partial z + \tau\partial t + \eta\partial T$

If $T^* T^*_{t^*} = T^*_{z^*z^*}$ (16)

Substituting the formulas in eqs.(12) into eq.(16) and replacing $T^*_{t^*}, T^*_{z^*z^*}$ wherever occurs ,

$(T + \varepsilon\eta + O(\varepsilon^2))(T_t + \varepsilon\eta_t + O(\varepsilon^2)) =$

$T_{zz} + \varepsilon\eta_{zz} + O(\varepsilon^2)$

Yields to the equation $T\eta_t + T_t\eta = \eta_{zz}$

Now Substituting the formulas in eqs.(13,14,15) in the above equation we are left with a polynomial equation involving the various derivatives of T whose coefficients are certain derivatives of ξ, τ and η only depend on z,t and T, we can equate the individual coefficients to zero ,leading to the complete set of *determining equations* which shown in table (1) :

by solving set of determining equation we obtain

$\xi = c_1 z + c_2$, $\tau = c_3$, $\eta = 0$

The vector fields span where ci are arbitrary constants then the symmetry algebra of the expulsion equation $v_1 = \partial t$, $v_2 = z\,\partial z$, $v_3 = \partial z$

2.2- Invariant of the Thermal Expulsion Equation under Stretching Group

If the PDE in eq . (11) in [8] invariant under the family in eq. (7) with $M = 1, N = -1, L = -2$ i.e. $\alpha - \beta = -2$

And if we substituting eq (10) into eq. (11). We will first obtain the partial derivatives,

$$T_t = \frac{\alpha}{\beta} t^{\frac{\alpha}{\beta}-1} y(x) + t^{\frac{\alpha}{\beta}} \left(-\frac{1}{\beta} z t^{\frac{-1}{\beta}-1}\right)\dot{y}$$

Then $T_t = t^{\frac{\alpha}{\beta}-1} \dfrac{(\alpha y - x\dot{y})}{\beta}$, $x = z\,t^{-1/\beta}$, $\dot{y} = \dfrac{dy}{dx}$

and $T_z = t^{\frac{\alpha}{\beta}} t^{\frac{-1}{\beta}} \dot{y} = t^{\frac{\alpha-1}{\beta}} \dot{y}$, $Tzz = t^{(\alpha-2)/\beta}\, \ddot{y}$, where $\ddot{y} = \dfrac{d\dot{y}}{dx}$

By substituting these partial derivatives in eq(11),we obtain

$\beta\,\ddot{y} + xy\,\dot{y} - \alpha\,y = 0$ (17)

this equation is the principal (ODE). Now

1- when α = - 1/2, β = 3/2 in [5], the eq. (12) becomes

$3\,\ddot{y} + x\,y^2 = 0$

This equation can easily be integrated and we get

$y = 6 / (x^2 + c^2)$,where c is the constant of the integratio

2- The boundary and initial conditions; in [5], [7]

T (0,t) = 1, t>0; T (∞ , t) = 0 ; T(z ,0) = 0 , 0<z<∞

which correspond to clamp temperature problem.The condition T (0,t) = 1 requires α = 0, β = 2, then the eq. (12) becomes

$2\,\ddot{y} + x\,y\,\dot{y} = 0$ (18)

Table (1): The Set of Determining Equations of Symmetries of Expulsion Equation

coefficient	variable of the term	coefficient	variable of the term
$\eta_t = 0$	T	$\eta_T - 2\xi_z = 0$	T_{zz}
$\eta_t - \tau_t = 0$	TT_t	$-2\tau_z = 0$	T_{tz}
$\xi_t = 0$	TT_z	$\eta_{TT} - 2\tau_{zT} = 0$	T_z^2
$\tau_t = 0$	TT_t^2	$-\tau_{zT} = 0$	$T_t T_z$
$\xi_T = 0$	$TT_t T_z$	$\xi_{TT} = 0$	T_z^3
$\eta + \tau_{zz} = 0$	T_t	$\tau_{TT} = 0$	$T_t T_z^2$
$\eta_{zz} = 0$	1	$-3\xi_T = 0$	$T_z T_{zz}$
$2\eta_{zt} - \xi_{zz} = 0$	T_z	$-\tau_T = 0$	$T_t T_{zz}$

With the boundary condition $y(0) = 1$, $y(\infty) = 0$
and the similarity solution has the form $T(z,t) = y(x)$, $x = z / t^{1/2}$
when $T(0,t) = F(t)$, where F(t) is a slowly varying function of time ,we try the solution of the form $T(z,t) = F(t)\, y(x)$, $x = z / p(t)$ (19)
where y(x) is the function determined by eq. (18) that satisfies the boundary conditions $y(0)=1$ and $y(\infty)=0$, and P(t) is a function yet to be determined , now since

$$\frac{d}{dt}\left(\frac{T^2}{2}\right) = TT_t$$

if we integrate eq.(11) with respect to z from zero to infinity ,we will find

$$\frac{d}{dt}\int_0^\infty \frac{T^2}{2} dz = -T_z(0,t) \qquad (20)$$

if we substitute eq. (19) in to eq.(20), we will find

$$\frac{d}{dt}\left[F^2(t)p(t)\int_0^\infty \frac{y^2}{2} dx\right] = \frac{-F(t)\dot{y}(x)}{p(t)}$$

(21) Now from eq (18) we find that

$$0 = 2\dot{y}]_0^\infty + \int_0^\infty xy\dot{y}dx = -2\dot{y}(0) - \int_0^\infty \frac{y^2}{2} dx$$

Where the integral $\int_0^\infty xy\dot{y}dx$ (the second equality) is obtained by integration by parts

we find that $\int_0^\infty (y^2/2)\, dx = 2$ since $y(0) = 1$, $y(\infty) = 0$ of eq.(18)

We find that $\int_0^\infty (y^2/2)\, dx = 2$ since $y(0) = 1$, $y(\infty) = 0$ of eq.(18)

Thus

$$\frac{d}{dt}(2F^2(t)P(t)) = \frac{F(t)}{P(t)}$$

, if we multiply this equation by $P(t)\, F^2(t)$ and integration we will find

$$\frac{2(F^2(t)P(t))^2}{2} = \int_0^t F^3(s)ds$$

, then

$$P(t) = \frac{\left[\int_0^t F^3(s)\,ds\right]^{1/2}}{F^2(t)} \qquad (22)$$

Then from eq.(19) $T_z(0,t) = \dfrac{F(t)\dot{y}(0)}{P(t)}$

(23)

Then by substituting eq. (22)in eq.(23) we obtain

$$T_z(0,t) = \frac{F^3(t)\,\dot{y}(0)}{\left[\int_0^t F^3(s)\,ds\right]^{1/2}}$$

3- The boundary and initial conditions in [5]

$T_z(0,t) = 1,\ T(\infty, t) = 0,\ T(z, 0) = 0$

Correspond to clamp heat flux problem requires that $\alpha = 1$, $\beta = 3$ then the similaritysolution

$$T(z,t) = t^{1/3}\,y(x),\quad x = \frac{z}{t^{1/3}}$$

Then the principal (ODE) in eq.(17) becomes $3\ddot{y} + xy\dot{y} - y^2 = 0$ (24)

with the boundary condition $\dot{y}(0) = -1$, $y(\infty) = 0$ Now when the heat flux is a slow varying function with respect to t $T_z(0,t) = -G(t)$

we try the solution of the form $T(z,t) = G(t)\,p(t)\,y(x)$, $x = z/p(t)$ (25)

As before we substitute eq. (25) in eq. (20)

$$\frac{d}{dt}\left[G^2(t)P^3(t)\int_0^\infty \frac{y^2}{2}dx\right] = G(t) \qquad (26)$$

By integrating eq.(26)from zero to infinity and then integrate by parts we find so that

$$\frac{d}{dt}\left[G^2(t)P^3(t)\right] = G(t)$$ or

$$P(t) = \left[\frac{1}{G^2(t)}\int_0^t G(s)\,ds\right]^{1/3}$$

and $$T(0,t) = y(0)\left[G(t)\int_0^t G(s)\,ds\right]^{1/3}$$

$$\int_0^\infty (y^2/2)\,dx = 1$$

3- Discussion

Classical Lie symmetry method is a new and powerful tool for constructing reduction for partial differential equation. If we know a symmetry group for PDE, by using algebra symmetry of PDE, we can calculate the similarity solution by solving the related (ODE) rather than PDE it self. The substitution of trial solution such as in eq (19,25) made up of group invariant reduce the PDE to an ODE, and help us for reduction the boundary value problem with PDE to boundary value problem with (ODE), and to find some information about $T(0,t)$ using a slow varying in it with respect to t.

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هاوشێوەکان وسۆلیۆشنە نەگۆڕەکان بۆ هاوکێشەی Thermal Expulsion

مها فالح جاسم

بەشی کومبیوتەر / پەیمانگای تەکنیکی / کەرکوک / هەرێمی کوردستان - عێراق

پوختە

لە م کوێژنیە وە دا دیراسەی هاوشێوەی (لی _ Lie) کلاسیکی بۆهاوکێشەی پارشنالیٚ جیاکاری دوو دایمێنشن (PDE) ئەکەین کە پیٚ ی ئە ووترێت هاوکێشەی (Termal Expulsion) ، کە دەستمان ئەکە وێت لە نێو رێگای هاوشێوەی گروپی (Lie) .

وە لەمە دوور تر ، کە م کردنە وە کانی ئە م هاوکێشە یە بۆهاوکێشەی جیاکاری ئاسایی لە(پلەی دووەمODE) مان دەست کەوت وە ناونرا بە هاوکیشەی جیاکاری ئاسایی بنەڕەتی لە نێونە گۆڕانی ئەم هاوکێشەیە لەژێر گروپی (One-Parameter Stretching) .

وەپاشان هەستاین بەش کردنەی کەم کردنە وەکان(ODEs) لەگەڵ مە رجەسنووری یەکانی لەکاتتیکداکەئەم مەرجانە جیٚ گیرنە بێت وەهە روەها کاتیك گۆڕانی هێواشی تیاڕوونە دات بۆ ئشتیقاقی سۆلیۆشنی نزیك بۆهاوکیشەی (Thermal Expulsion) بە پیٚ ی مە رجە سنووری یەکانی.

التناظر والحلول اللامتغايرة لمعادلة الطرد الحرارية

مها فالح جاسم

قسمالكومبيوتر / المعهد التقني / كركوك / اقليم كردستان - العراق

الخلاصة

في هذا البحث ندرس طريقة تناظر لي الاعتيادي(الكلاسيكي) لمعادلة تفاضلية جزئية(PDE) ثنائية الأبعاد والتي تسمى معادلة الطرد الحرارية. وحصلنا على اختزالات إلى معادلة تفاضلية اعتيادية من الرتبة الثانية (ODE) وسميت معادلة تفاضلية اعتيادية أساسيه.

ثم حللنا بعض مسائل معادلة الطرد الحرارية عندما تكون المعادلة لا متغايرة تحت زمرة الشد لاشتقاق حلول تقريبية لمعادلة الطرد الحرارية طبقا لشروطها الحدودية عند ثبوت هذه الشروط مرة وعند حدوث تغير بطيء فيها بالنسبة للزمن مرة ثانية.

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