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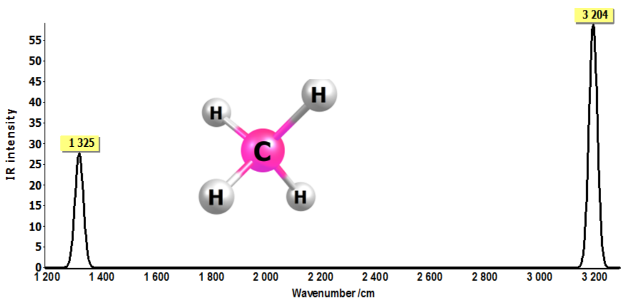

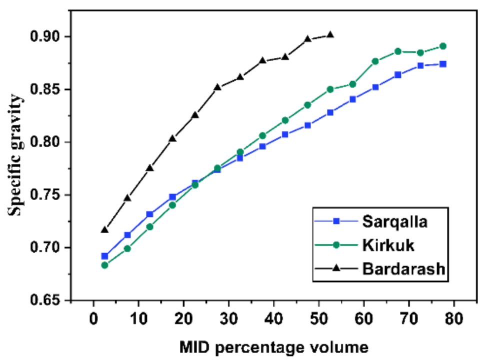

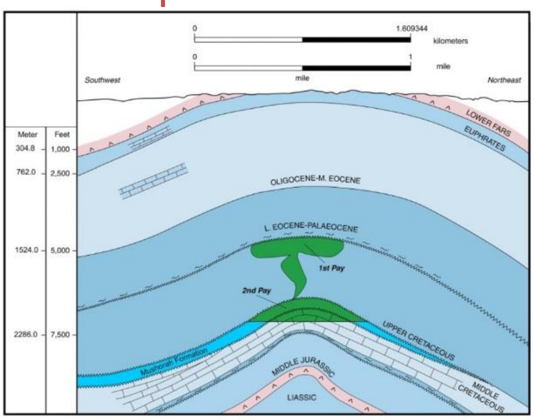

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Krull's Principal Ideal Theorem for Locally Noetherian Rings

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Abstract

A ring R is Locally Noetherian if $R_{\mathcal{P}}$ is Noetherian for each prime ideals $\mathcal{P}$ of R. In this paper we study Locally Noetherian rings. We show that Krull's Principal Ideal Theorem and Generalized Principal Ideal Theorem are also true for Locally Noetherian rings. In general, Locally Noetherian rings do not have finitely many minimal prime ideals , a sufficient condition is given under which they have finitely many minimal prime ideals.

Introduction

Throught this paper we let R to be a commutative ring with identity and $\mathcal{S}$ be a multiplicative system in R. A localization of R at the $\mathcal{S}$ denoted by $R_{\mathcal{S}} = \left\{\frac{a}{s}, a \in R, s \in \mathcal{S}\right\}$ (see[1, 10]). For a prime ideal $\mathcal{P}$ of R the set $\mathcal{S} = R - \mathcal{P}$ is a multiplicative system in R and we shall write $R_{\mathcal{P}}$ for the localization of R at $R - \mathcal{P}$. A ring R is called a Locally Noetherian ring if $R_{\mathcal{P}}$ is Noetherian for all prime ideals $\mathcal{P}$ of R (see [1]), R is Locally Noetherian if $R_{\mathcal{M}}$ is Noetherian for each maximal ideals $\mathcal{M}$ of R (see [6, 8]). Every Noetherian ring R is Locally Noetherian [1]. However, the converse is not true in general, it is shown in the first section . Some of the motivation for studying Noetherian rings generalizes to Locally Noetherian rings, for example Hilbert's Basis Theorem. In this work we are interested in extending some properties of ideals to Locally Noetherian rings. Indeed the rest of this paper is devoted to extend Krull's Principal Ideal Theorem of Noetherian rings to Locally Noetherian rings and some further properties of Locally Noetherian rings have been proved. Also, we introduce the $\mathcal{S}$-height of an ideal I. The spectrum of R is the set of all prime ideals of R and denoted by $Spec(R)$, and also $\mathcal{S}Spec(R) = \{\mathcal{P}: \mathcal{P} \in Spec(R)\ such\ that\ \mathcal{P} \cap \mathcal{S} = \emptyset\}$ (see [9]). We write $Max(R)$ for the maximal spectum of R (see [10]). A $\mathcal{P} \in Spec(R)$ is a minimal prime of I if $I \subseteq \mathcal{P}$ and R contains no other prime ideal $\mathcal{Q}$ with $I \subseteq \mathcal{Q} \subsetneq \mathcal{P}$, we shall write $Min(I)$ for the set of all minimal primes of I. Further, $\mathcal{P} \in$

$Spec(R)$ is a minimal prime ideal of R if $\mathcal{P} \in Min(\langle 0 \rangle)$ (see [10]). A prime ideal $\mathcal{P}$ of R is an $\mathcal{S}$-minimal prime ideal of an ideal I of R, if $\mathcal{P}$ is minimal among the prime ideals which contain I and trivially meet $\mathcal{S}$ (see [9]), similarly we write $\mathcal{S}Min(I)$ for the set of all $\mathcal{S}$-minimal prime ideal of I. Unlike Noetherian rings Locally Noetherian rings may not have finitely many prime ideals in general, see Example 2.22. In Theorem 2.25 we show that any ideal in a Locally Noetherian ring has finitely many $\mathcal{S}$-minimal prime ideals. In [10] the hight of $\mathcal{P} \in Spec(R)$ is defined by

$$ht(\mathcal{P}) = \sup\{n \in \mathbb{N}: \mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots \subsetneq \mathcal{P}_n = \mathcal{P},\ with\ \mathcal{P}_i \in Spec(R)\}.$$

and the height of I, denoted by $ht(I)$, is the minimum of the height of the prime ideals containing I. .

Some basic definitions and results

In this section we recall some basic definitions and results which are used in this paper. First, we start with the following examples

Example 1.1 *Consider $P(\mathbb{N})$, the power set of natural numbers $\mathbb{N}$.(+) and (.) defined on $P(\mathbb{N})$ by: $A + B = A \cup B - A \cap B$ and $A \cdot B = A \cap B$, for all $A, B \in P(\mathbb{N})$. Then $P(\mathbb{N})$ is Locally Noetherian but not Noetherian* .

Example 1.2 Let $R = \prod_{i\in\mathbb{N}} \mathbb{F}$, where $\mathbb{F}$ is a field. Then the ring R is not Noetherian, since $\{(\alpha_1, 0,0,0, \cdots): \alpha_1 \in \mathbb{F}\} \subseteq \{(\alpha_1, \alpha_2, 0,0, \cdots): \alpha_1, \alpha_2 \in \mathbb{F}\} \subseteq \{(\alpha_1, \alpha_2, \alpha_3, 0,0, \cdots): \alpha_i \in \mathbb{F}\} \subseteq \cdots$ is an ascending chain of ideals of R which does not terminate. However, R is Locally Noetherian. To show this, for any $(\alpha_1, \alpha_2, \alpha_3, \cdots) \in R$, there exists $(\beta_1, \beta_2, \beta_3, \cdots) \in R$, where if $\alpha_i \neq 0$,then $\beta_i = {\alpha_i}^{-1}$ and if $\alpha_i = 0$, then $\beta_i = 0$, therefore $(\alpha_1, \alpha_2, \alpha_3, 0, 0, \cdots)(\beta_1, \beta_2, \beta_3, ...)(\alpha_1, \alpha_2, \alpha_3, 0,0, \cdots) = (\alpha_1, \alpha_2, \alpha_3, 0,0, \cdots)$. Hence R is von Neumann regular ring, see[2]. Thus all prime ideals in R are maximal. Further $R_{\mathcal{P}}$ is a field for each $\mathcal{P} \in Spec(R)$. Therefore R is Locally Noetherian.

Hilbert's Basis Theorem states that a commutative ring R is Noetherian if and only if $R[X]$ is Noetherian, see [10]. In fact this theorem generalizes to Locally Noetherian rings, see [8]. Starting with a Locally Noetherian ring that is not Noetherian this generalization provides a class of examples of Locally Noetherian rings which are not Noetherian.

Definition 1.3 *[12] Let R be an integral domain. Then R is a Dedekind domain if every non zero ideal of R is invertible, and it is an almost Dedekind domain if $R_{\mathcal{M}}$ is a Dedekind domain for each $\mathcal{M} \in Max(R)$.*

Lemma 1.4 *Every almost Dedekind domain is Locally Noetherian.*

The converse of Lemma 1.4 is not true in general, the rings in Examples 1.1 and 1.2 are Locally Noetherian but not almost Dedekind domain since each of them is not a domain.

Using Lemma 1.4, one can produce more examples of Locally Noetherian rings which are not Noetherian.

Examples 1.5 (i) Let p be a prime number and ζ_p be a primitive p^{th} root of unity. The ring $R = \mathbb{Z}[\zeta_2, \zeta_3, ..., \zeta_P, ...]$ is almost Dedekind. Hence, it is Locally Noetherian, but not Noetherian, see [4].

(ii) Consider the ring $R = \mathbb{Z}[\sqrt{2}, \sqrt{3}, ..., \sqrt{\mathrm{p}_n}, ...]$ where p_n is the n^{th} prime number in $\mathbb{Z}$. The integral closure of R is a non Noetherian Locally Noetherian domain, see [5].

Let D be a domain and $\mathfrak{S}$ be a semigroup. We shall write $D[\mathfrak{S}]$ for the semigroup ring of the semigroup $\mathfrak{S}$ with base ring D. Then $D[\mathfrak{S}]$ is an almost Dedekind domain under certain conditions due to Gilmer and Parker in [6], which we recall in the next theorem.

Theorem 1.6 [7, Theorem 36] *Let $\mathfrak{S}$ be an additive, torsion-free, cancellative, semigroup. Then $D[\mathfrak{S}]$ is an almost Dedekind domain if and only if D is a field and one of the following holds:*

1. we have $(\mathfrak{S}, +) \simeq (\mathbb{Z}_{\geq 0}, +)$ as monoids.

2. $char(D) = 0$ and $\mathfrak{S} \simeq$ G as groups, for some additive subgroup G of $\mathbb{Q}$ which contains $\mathbb{Z}$.

3. $char(D) = q > 0$ and $\mathfrak{S} \simeq$ G as groups, for some additive subgroup G of $\mathbb{Q}$ which contains the set $\{q \in \mathbb{Z} \mid \frac{1}{q^k} \notin \mathfrak{S}$ for some $k\}$.

Lemma 1.7 [9]. *Let $\mathcal{S}$ be a multiplicative system in R. If $a \in R$ and $s \in \mathcal{S}$, then $\langle a \rangle_{\mathcal{S}} = \left\langle \frac{a}{s} \right\rangle$.*

The following Lemma is a straightforward generalisation of Lemma 2.1, which we shall record its statement for later use.

Lemma 1.8 *If $a_1, a_2, \cdots, a_n \in R$ and $s_1, s_2, \cdots, s_n \in \mathcal{S}$, then*

$$\langle a_1, a_2, \cdots, a_n \rangle_{\mathcal{S}} = \left\langle \frac{a_1}{s_1}, \frac{a_2}{s_2}, \cdots, \frac{a_n}{s_n} \right\rangle.$$

Proposition 1.9 [12] *For an ideal I of R we have $I \cap \mathcal{S} \neq \emptyset$ if and only if $I_{\mathcal{S}} = R_{\mathcal{S}}$, (or equivalently $I \cap \mathcal{S} = \emptyset$ if and only if $I_{\mathcal{S}} \neq R_{\mathcal{S}}$).*

Corollary 1.10 [12]. *The prime ideals of R which do not meet a multiplicative system $\mathcal{S}$ are in bijection with the proper prime ideals of $R_{\mathcal{S}}$.*

Theorem 1.11 *[12] Let $\mathcal{P}_1, \mathcal{P}_2 \in Spec(R)$ such that $\mathcal{P}_1 \cap \mathcal{S} = \emptyset = \mathcal{P}_2 \cap \mathcal{S}$. Then $\mathcal{P}_1 = \mathcal{P}_2$ if and only if $(\mathcal{P}_1)_{\mathcal{S}} = (\mathcal{P}_2)_{\mathcal{S}}$.*

Corollary 1.12 *Let $\mathcal{P}_1, \mathcal{P}_2 \in Spec(R)$ with $\mathcal{P}_1 \cap \mathcal{S} = \emptyset = \mathcal{P}_2 \cap \mathcal{S}$. Then there is no prime ideal strictly lies between $\mathcal{P}_1$ and $\mathcal{P}_2$ if and only if there is no prime ideal strictly lies between $(\mathcal{P}_1)_{\mathcal{S}}$ and $(\mathcal{P}_2)_{\mathcal{S}}$.*

Corollary 1.13 [9]. *Let $\mathcal{P} \in Spec(R)$. If I is an ideal of R and $Q \in Min(I)$ with $Q \subseteq \mathcal{P}$, then $Q_{\mathcal{P}} \in Min(I_{\mathcal{P}})$.*

Theorem 1.14 [10]. *Let R be a commutative Noetherian ring and let a* G *R be a non-unit. Let P be a minimal prime ideal of the principal ideal aR of R. Then htP* < 1.

Theorem 1.15 [10]. *Let R be a commutative Noetherian ring and let I be a proper ideal of R which can be generated by n elements. Then htP<n for each minimal prime ideal P of I.*

Theorem 1.16 [11]. *Let R be a Noetherian ring, I an ideal in R generated by n elements, $I \neq R$. Let P be a prime ideal containing I. Assume that the rank of P/I in the ring R/I is k. Then the rank of P in* R *is at most n + k.*

Corollary 1.17 [[9, Corollary 2.2.15]. *Let $\mathcal{P}, \mathcal{Q} \in Spec(R)$, and I be an ideal of R. If $\mathcal{Q}_{\mathcal{P}} \in Min(I_{\mathcal{P}})$, then $\mathcal{Q} \in \mathcal{S}Min(I)$.*

Corollary 1.18 [9, Corollary 2.2.19]. *Let $\mathcal{P} \in Spec(R)$, and I be an ideal of R. Then, there is bijection between $\mathcal{S}Min(I)$ and $Min(I_{\mathcal{P}})$.*

Theorem 1.19 [13, Theorem 12.3]. Any ideal of a Noetherian ring has a finite number of minimal prime ideals.

Theorem 1.20 [10, Theorem 3.61]. *Let $P_1, P_2, \dots, P_n$, where $n > 2$, be ideals of the commutative ring R such that at most 2 of $P_1, P_2, \dots, P_n$ are not prime. Let S be an additive subgroup of R which is closed under multiplication. (For example,S could be an ideal of R, or a subring of R). Suppose that $S \subseteq \bigcup_{i=1}^{n} P_i$. Then $S \subseteq P_j$ for some j with $1 < j < n$.*

Corollary 1.21 [10, Corollary 15.15]. *Let R be a commutative Noetherian ring, and let I be a proper ideal of R which can be generated by n elements. Let $P \in Spec(R)$ be such that $I \subseteq P$. Then $ht_{\frac{R}{I}}\frac{P}{I} \le ht_R P \le ht_{\frac{R}{I}}\frac{P}{I} + n$.*

Main Results

Proposition 2.1.*Let $I_i \in Spec(R)$, for each $1 \le i \le m$ such that $I_i \cap \mathcal{S} = \emptyset$. Then $I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_n$ if and only if $(I_1)_{\mathcal{S}} \subsetneq (I_2)_{\mathcal{S}} \subsetneq \cdots \subsetneq (I_n)_{\mathcal{S}}$.*

Proof. The proof follows by applying the principal of induction on Theorem 1.11 .

We recall the following standard result regarding the relation between the $ht(\mathcal{P})$ and $ht(\mathcal{P}_{\mathcal{S}})$, we include the proof for the sake of self-containment.

Lemma 2.2 *For each $\mathcal{P} \in Spec(R)$ with $\mathcal{P} \cap \mathcal{S} = \emptyset$, we have $ht(\mathcal{P}_{\mathcal{S}}) \le n$ if and only if $ht(\mathcal{P}) \le n$.*

Proof. Let $ht(\mathcal{P}_{\mathcal{S}}) \le n$. To show $ht(\mathcal{P}) \le n$. If $ht(\mathcal{P}) > n$, then there exists $m > n$ such that

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_m = \mathcal{P}$$

is a chain of prime ideals in R. Since $\mathcal{P}_i \subsetneq \mathcal{P}$ for each $1 \le i \le m$ and $\mathcal{P} \cap \mathcal{S} = \emptyset$, therefore $\mathcal{P}_i \cap \mathcal{S} = \emptyset$. Then by Corollary 1.10 and Proposition 2.1 the above chain gives the following chain of prime ideals in $R_{\mathcal{S}}$,

$$(\mathcal{P}_0)_{\mathcal{S}} \subsetneq (\mathcal{P}_1)_{\mathcal{S}} \subsetneq \cdots \subsetneq (\mathcal{P}_m)_{\mathcal{S}} = \mathcal{P}_{\mathcal{S}}.$$

Hence $ht(\mathcal{P}_{\mathcal{S}}) \ge m > n$, which is contradiction. Hence $ht(\mathcal{P}) \le n$.

Conversely, assume that $ht(\mathcal{P}) \le n$, . Suppose that $ht(\mathcal{P}_{\mathcal{S}}) > n$ which means there is an $m > n$ such that

$$\mathcal{Q}_0 \subsetneq \mathcal{Q}_1 \subsetneq \cdots \subsetneq \mathcal{Q}_m = \mathcal{P}_{\mathcal{S}}$$

is a chain of prime ideals $\mathcal{Q}_i$ in $R_{\mathcal{S}}$. By Corollary 1.10 for each $1 \le i \le m$, we have $\mathcal{Q}_i = (\mathcal{P}_i)_{\mathcal{S}}$, for the prime ideals $\mathcal{P}_i = \left\{x \in R : \frac{x}{1} \in \mathcal{Q}_i\right\}$ with $\mathcal{P}_i \cap \mathcal{S} = \emptyset$, therefore

$$(\mathcal{P}_0)_{\mathcal{S}} \subsetneq (\mathcal{P}_1)_{\mathcal{S}} \subsetneq \cdots \subsetneq (\mathcal{P}_m)_{\mathcal{S}} = \mathcal{P}_{\mathcal{S}}.$$

Since $\mathcal{P}_i \cap \mathcal{S} = \emptyset$, by Proposition 2.1 the above chain gives the following chain of prime ideals $\mathcal{P}_i$ in R,

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \cdots \subsetneq \mathcal{P}_n = \mathcal{P}$$

that means $ht(\mathcal{P}) \ge m > n$, which is contradiction. Hence $ht(\mathcal{P}_{\mathcal{S}}) \le n$.

Corollary 2.3 *Let $\mathcal{P} \in Spec(R)$ such that $\mathcal{P} \cap \mathcal{S} = \emptyset$. Then $ht(\mathcal{P}_{\mathcal{S}}) = n$ if and only if $ht(\mathcal{P}) = n$.*

Proof. The proof follows from Lemma 2.2 and Corollary 1.12 .

Krull's principal ideal theorem states that in a Noetherian ring the height of minimal prime ideals over a principal ideal is at most one, see [10]. We generalizes this theorem to Locally Noetherian rings in the following theorem.

Theorem 2.4 *(Krull's Principal Ideal Theorem) Let R be a Locally Noetherian ring and I be a principle ideal of R. If $\mathcal{P} \in \mathrm{Min(I)}$, then $\mathcal{P}$ has height at most one.*

Proof. By Corollary 1.13 we get that $\mathcal{P}_{\mathcal{P}}$ is a minimal prime ideal over $I_{\mathcal{P}}$ of $R_{\mathcal{P}}$. As I is a principal ideal say $I = \langle a \rangle$, by Lemma 1.7 we get that $I_{\mathcal{P}} = \langle a \rangle_{\mathcal{P}} = \left\langle \frac{a}{p} \right\rangle$ for some $p \notin \mathcal{P}$. Therefore $R_{\mathcal{P}}$ is Noetherian and $\mathcal{P}_{\mathcal{P}}$ is a minimal over $I_{\mathcal{P}}$ of $R_{\mathcal{P}}$. Then by Theorem 1.14 we have $\mathcal{P}_{\mathcal{P}}$ has height at most one, that is $ht(\mathcal{P}_{\mathcal{P}}) \leq 1$. By Lemma 2.2 we get that $ht(\mathcal{P}) \leq 1$.

Corollary 2.5 *Let R be a Locally Noetherian ring and $a \in R$. If a is a non-zero divisor and $\mathcal{P} \in \mathrm{Min}(\langle a \rangle)$ then $ht(\mathcal{P}) = 1$.*

Proof. By Theorem 2.4 we have $ht(\mathcal{P}) \leq 1$. Now if $ht(\mathcal{P}) = 0$, then $\mathcal{P} \in Min(\langle 0 \rangle)$. Hence it consists of zero divisors, and so a is a zero divisor. Thus $ht(\mathcal{P}) = 1$.

Corollary 2.6 *Let R be a Locally Noetherian ring and let $x \in R$. If $\mathcal{P}, \mathcal{Q} \in Spec(R)$ and $\mathcal{Q} \in \mathrm{Min}(\langle \mathcal{P}, x \rangle)$, then there is no prime strictly between $\mathcal{P}$ and $\mathcal{Q}$.*

Proof. Theorem 2.4 implies that $\frac{\mathcal{Q}}{\mathcal{P}}$ is prime of height 1 or 0 in $\frac{R}{\mathcal{P}}$. Hence there can not be a prime strictly between $\mathcal{P}$ and $\mathcal{Q}$.

Theorem 2.7 *Let R be a Locally Noetherian ring and $I = < a_1, a_2, \ldots, a_n >$ is a proper in R. , If $\mathcal{P} \in \mathrm{Min(I)}$, then $\mathcal{P}$ has height at most n.*

Proof. Let $\mathcal{P} \in \mathrm{Min}(I)$ in R. Then by Corollary 1.13 we get that $\mathcal{P}_{\mathcal{P}} \in Min(I_{\mathcal{P}})$ in $R_{\mathcal{P}}$.Then Lemma 1.8 implies that $I_{\mathcal{P}} = \langle a_1, a_2, \cdots, a_n \rangle_{\mathcal{P}} = \left\langle \frac{a_1}{p_1}, \frac{a_2}{p_2}, \cdots, \frac{a_n}{p_n} \right\rangle$, for some $p_1, p_2, \ldots, p_n \notin \mathcal{P}$. The ideal $I_{\mathcal{P}}$ is proper and generated by n elements in $R_{\mathcal{P}}$, since $I \subseteq \mathcal{P}$. Therefore $R_{\mathcal{P}}$ is Noetherian and $\mathcal{P}_{\mathcal{P}} \in Min(I_{\mathcal{P}})$. Hence by Theorem 1.15, we have $ht(\mathcal{P}_{\mathcal{P}}) \leq n$. Finally, we have $ht(\mathcal{P}) \leq n$ by Lemma 2.2 .

Corollary 2.8 *Let R be a Locally Noetherian ring and let $x_1, \ldots, x_n \in R$. If $\mathcal{P}, \mathcal{Q} \in Spec(R)$ and $\mathcal{Q} \in \mathrm{Min}(\langle \mathcal{P}, x_1, \ldots, x_n \rangle)$, then every chain of primes between $\mathcal{P}$ and $\mathcal{Q}$ has length at most n.*

Proof. The proof follows from Theorem 2.7.

Proposition 2.9. *If R is a Locally Noetherian ring and I is a proper ideal generated by n elements of R, then each $\mathcal{P} \in Min(I)$ we have $ht(I) \leq n$. Further, if $ht_{\frac{R}{I}}\left(\frac{\mathcal{P}}{I}\right) = k$, then $ht_R(\mathcal{P}) \leq n + k$.*

Proof. The proof follows from Proposition 1.9 and Theorem 1.16 [11].

Theorem 2.10 *Every prime ideals in a Locally Noetherian ring has finite height.*

Proof. Let $\mathcal{P} \in Spec(R)$. Then $R_{\mathcal{P}}$ is Noetherian and $\mathcal{P}_{\mathcal{P}} \in Spec(R_{\mathcal{P}})$. By ([10], Corollary 2.6) we get that $\mathcal{P}_{\mathcal{P}}$ has finite height, say n. We claim that $ht(\mathcal{P}) \leq n$. If we assume that $ht(\mathcal{P}) = m > n$, then there is a chain of prime ideals

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_m = \mathcal{P}$$

in R. For each $1 \leq i \leq m$ we have $\mathcal{P}_i \cap (R - \mathcal{P}) = \emptyset$, since $\mathcal{P}_i \subsetneq \mathcal{P}$. Then by Corollary 1.10 and Proposition 2.1 the above chain gives the following chain of prime ideals $(\mathcal{P}_i)_{\mathcal{P}}$ in $R_{\mathcal{P}}$,

$$(\mathcal{P}_0)_{\mathcal{P}} \subsetneq (\mathcal{P}_1)_{\mathcal{P}} \subsetneq \cdots \subsetneq (\mathcal{P}_m)_{\mathcal{P}} = \mathcal{P}_{\mathcal{P}}.$$

Hence $ht(\mathcal{P}_{\mathcal{P}}) \geq m > n$ which is contradiction. The result follows.

Proposition 2.11 *Let R be a Locally Noetherian ring, and let $\mathcal{P}, \mathcal{Q} \in Spec(R)$ with $\mathcal{P} \subseteq \mathcal{Q}$. Then $ht(\mathcal{P}) \leq ht(\mathcal{Q})$, and $ht(\mathcal{P}) = ht(\mathcal{Q})$ if and only if $\mathcal{P} = \mathcal{Q}$.*

Proof. Let $ht(\mathcal{P}) = n$, then there exists a chain

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_n = \mathcal{P} \subseteq \mathcal{Q}$$

of prime ideals of R. Obviously, $ht(\mathcal{P}) \leq ht(\mathcal{Q})$. Now suppose that $ht(\mathcal{P}) = ht(\mathcal{Q}) = n$. If $\mathcal{P} \neq \mathcal{Q}$ the chain

$$\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_n \subsetneq \mathcal{Q}$$

of prime ideals of R implies that $ht(\mathcal{Q}) > n$, which contradicts $ht(\mathcal{Q}) = ht(\mathcal{P}) = n$. Hence $\mathcal{P} = \mathcal{Q}$.

Proposition 2.12 *A Locally Noetherian ring R satisfies the decending chain condition on prime ideals.*

Proof. Let $\mathcal{P}_0 \supsetneq \mathcal{P}_1 \supsetneq \mathcal{P}_2 \supsetneq \cdots$ be a strictly descending chain of prime ideals in R. Then by Theorem 2.10 the height of $\mathcal{P}_0$ is finite. Thus the chain terminates.

Proposition 2.13. *Let $I = \langle a_1, a_2, \dots, a_n \rangle$ be an ideal in a Locally Noetherian ring R. Then $ht(I) \leq n$.*

Proof. Let $\mathcal{P} \in Spec(R)$ with $I \subseteq \mathcal{P}$. Then by Lemma 1.8 we get that $I_{\mathcal{P}} = \left\langle \frac{a_1}{q_1}, \frac{a_2}{q_2}, \dots, \frac{a_n}{q_n} \right\rangle$, for some $q_i \notin \mathcal{P}$. From Remark 15.7 in [10], we get $ht(I_{\mathcal{P}}) \leq n$. Therefore there is $\mathcal{Q} \in Spec(R_{\mathcal{P}})$ with $ht(\mathcal{Q}) = m \leq n$ with $I_{\mathcal{P}} \subseteq \mathcal{Q}$. This yields that there is a chain of prime ideals

$$\mathcal{Q}_0 \subsetneq \mathcal{Q}_1 \subsetneq \cdots \subsetneq \mathcal{Q}_m = \mathcal{Q}$$

in $R_{\mathcal{P}}$. By Corollary 1.10 for each $1 \leq i \leq m$ we have $\mathcal{Q}_i = (K_i)_{\mathcal{P}}$, for the prime ideals $K_i = \left\{ x \in R : \frac{x}{1} \in \mathcal{Q}_i \right\}$ with $K_i \cap (R - \mathcal{P}) = \emptyset$. Therefore

$$(K_0)_{\mathcal{P}} \subsetneq (K_1)_{\mathcal{P}} \subsetneq \cdots \subsetneq (K_m)_{\mathcal{P}} = K_{\mathcal{P}},\ with\ I_{\mathcal{P}} \subseteq K_{\mathcal{P}}.$$

Since $K_i \cap R - \mathcal{P} = \emptyset$, by Proposition 2.1 the above chain gives the following chain of prim ideals K_i in R,

$$K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_m = K,\ with\ I \subseteq K.$$

Hence $ht(K) \geq m$. If we assume that $ht(K) = t > m$, then there exists a chain

$$H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_t = K,$$

of prime ideals H_i in R. For each $1 \leq i \leq t$ which means $H_i \cap (R - \mathcal{P}) = \emptyset$ we have $K \cap (R - \mathcal{P}) = \emptyset$ and $H_i \subsetneq K$. Then by Corollary 1.10 and Propsition 2.1 the above chain gives the following chain of prime ideals $(H_i)_{\mathcal{P}}$ in $R_{\mathcal{P}}$,

$$(H_0)_{\mathcal{P}} \subsetneq (H_1)_{\mathcal{P}} \subsetneq \cdots \subsetneq (H_t)_{\mathcal{P}} = K_{\mathcal{P}}.$$

Hence $ht(Q) = ht(K_{\mathcal{P}}) \geq t > m$, which is contradiction. Hence $ht(K) = m$. Thus $ht(I) \leq m \leq n$.

Proposition 2.14 *Let R be a Locally Noetherian ring and let $I \subseteq \mathcal{P}$ be a chain of ideals of R such that $\mathcal{P}$ is prime. If $ht(I) = ht(\mathcal{P})$, then $\mathcal{P} \in \mathrm{Min}(I)$.*

Proof. Suppose that $\mathcal{P} \notin Min(I)$. Then there exists $Q \in Min(I)$ such that $I \subseteq Q \subset \mathcal{P}$. By Theorem 2.17 $\mathcal{P}$ and Q have finite height. Hence by Proposition 2.11 we have $ht(I) \leq ht(Q) < ht(\mathcal{P})$, which contradicts the fact that $ht(I) = ht(\mathcal{P})$. Hence $\mathcal{P} \in Min(I)$.

Recall that every ideal of a Noetherian ring has a finite number of minimal prime ideals, see [13]. However, this is not true in Locally Noetherian rings in general as the following example shows .

Example 2.15. From Example 1.2 we have $R = \prod_{i\in\mathbb{N}} \mathbb{F}$ is a Locally Noetherian ring which is not Noetherian. The following ideals are prime in R . Let $\mathcal{P}_i = \{(\alpha_0, \alpha_1, \alpha_2, \ldots): \alpha_1', \alpha_2', \alpha_3', \ldots \in \mathbb{F} \text{ and } \alpha_i = 0\}$. Further $\mathcal{P}_i \in \mathrm{Min}(\langle(0,0,0,\ldots)\rangle)$, for all i. Hence the ring R has an infinite number of minimal prime ideals. Let $\mathcal{S} = R - \mathcal{P}_0$. Then $\mathcal{P}_0$ is the only prime ideal does not meet $\mathcal{S}$. Hence the ring R has a finite number of $\mathcal{S}$-minimal prime ideals..

Next we show that for each ideal of a Locally Noetherian ring the set $\mathcal{S}Min(I)$ is finite.

Theorem 2.16. *Let R be a Locally Noetherian ring and I be an ideal of R. Then $\mathcal{S}Min(I)$ is finite.*

Proof. If $\mathcal{P} \in Spec(R)$, then $R_{\mathcal{P}}$ is a Noetherian ring and $I_{\mathcal{P}}$ is an ideal of $R_{\mathcal{P}}$. By Theorem 1.19, we have $Min(I_{\mathcal{P}}) = \{\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_n\}$ and for each $1 \leq i \leq n$, $\bar{Q}_i = (Q_i)_{\mathcal{P}}$, for the prime ideals $Q_i = \left\{x \in R: \frac{x}{1} \in \bar{Q}_\iota\right\}$. Then by Corollary 1.17 each $Q_i \in \mathcal{S}Min(I)$. Using Corollary 1.18 there is a one to one correspondence between $\mathcal{S}Min(I)$ and $Min(I_{\mathcal{P}})$. Hence I has a finite number of $\mathcal{S}$-minimal prime ideals.

Next, we introduce the following new concept. .

Definition 2.17 *Let I be a proper ideal of R. Then we define the $\mathcal{S}$-height of I by*

$$\mathcal{S}t(I) \coloneqq \min\{ht(\mathcal{P}): \mathcal{P} \in \mathcal{S}Spec(R) \text{ and } I \subseteq \mathcal{P}\}.$$

Theorem 2.18 *Let $\mathcal{S}$ be a multiplicative system in a Locally Noetherian ring R and let $\mathcal{P} \in \mathcal{S}Spec(R)$ with $\mathcal{S}t(\mathcal{P}) = n$. Then there is an ideal I of R which can be generated by n elements with $\mathcal{S}t(I) = n$ such that $I \subseteq \mathcal{P}$.*

Proof. To prove the theorem, we employ mathematical induction on n. If $n = 0$, then $\mathcal{P}$ has height 0. Hence it is minimal over $I = \{0\}$. Now let $\mathcal{S}t(\mathcal{P}) = n$, then there exists a chain $\mathcal{P}_0 \subsetneq \cdots \subsetneq \mathcal{P}_{n-1} \subsetneq \mathcal{P}_n = \mathcal{P}$ of prime ideals of R with $\mathcal{P} \cap \mathcal{S} = \emptyset$. Note that $\mathcal{S}t(\mathcal{P}_{n-1}) = n - 1$, since we have $\mathcal{S}t(\mathcal{P}_{n-1}) \geq n - 1$ but by Proposition 2.11 , $\mathcal{S}t(\mathcal{P}_{n-1}) < \mathcal{S}t(\mathcal{P}) = n$. Therefore we can apply the inductive hypothesis to $\mathcal{P}_{n-1}$. Hence there exists a proper ideal J of R which can be generated by $n - 1$ elements, say $J = \langle a_1, \ldots, a_{n-1}\rangle$, with $J \subseteq \mathcal{P}_{n-1}$ and $\mathcal{S}t(J) = n - 1$. By Proposition 2.14 we get that $\mathcal{P}_{n-1} \in \mathcal{S}Min(J)$. from Theorem 2.16 the ideal J has only finitely many $\mathcal{S}$-minimal prime ideals, and by Theorem 2.7 each of these $\mathcal{S}$-minimal have height exactly $n - 1$. Let the other $\mathcal{S}$-minimal prime ideals of J, in addition to $\mathcal{P}_{n-1}$ be $Q_1, Q_2, \ldots, Q_t$. Now we have $\mathcal{P} \nsubseteq \mathcal{P}_{n-1} \cup Q_1 \cup \ldots \cup Q_t$, otherwise Theorem 1.20 (see [10]) implies that $\mathcal{P} \subseteq \mathcal{P}_{n-1}$ or $\mathcal{P} \subseteq Q_i$ for some

$1 \le i \le t$, and none of these possibilities can occur since $\mathcal{S}t(\mathcal{P}) = n$ while $\mathcal{S}t(\mathcal{P}_{n-1}) = \mathcal{S}t(\mathcal{Q}_1) = \cdots = \mathcal{S}t(\mathcal{Q}_t) = n-1$. Therefore, there exists $a_n \in \mathcal{P} \setminus (\mathcal{P}_{n-1} \cup \mathcal{Q}_1 \cup \ldots \cup \mathcal{Q}_t)$. Define $I \coloneqq \sum_{i=1}^{n} Ra_i = J + Ra_n$. We show that I has all the required properties. It is clear from its definition that I can be generated by n elements and that $I = J + Ra_n \subseteq \mathcal{P}_{n-1} + \mathcal{P} = \mathcal{P}$. We show that $\mathcal{S}t(I) = n$. Since $J \subseteq I \subseteq \mathcal{P}$, $\mathcal{S}t(J) = n-1$, and $\mathcal{S}t(\mathcal{P}) = n$, we must have $\mathcal{S}t(I) = n-1$ or n. If $\mathcal{S}t(I) = n-1$, then there exists a $\mathcal{P}' \in \mathcal{S}Min(I)$ with $\mathcal{S}t(\mathcal{P}') = n-1$. Now $J \subseteq I \subseteq \mathcal{P}'$ and $\mathcal{S}t(J) = \mathcal{S}t(\mathcal{P}') = n-1$, therefore from Proposition 2.14 we have $\mathcal{P}'$ is one of the $\mathcal{S}Min(J)$. That is, $\mathcal{P}'$ is one of $\mathcal{P}_{n-1}, \mathcal{Q}_1, \ldots, \mathcal{Q}_t$, but this is not possible since $a_n \in I \subseteq \mathcal{P}'$ while a_n belongs to none of $\mathcal{P}_{n-1}, \mathcal{Q}_1, \ldots, \mathcal{Q}_t$, which is contradiction. Hence, we must have $\mathcal{S}t(I) = n$.

Theorem 2.19 *Let R be a Locally Noetherian ring, and let I be a proper ideal of R which can be generated by n elements. Let $\mathcal{P} \in Spec(R)$ such that $I \subseteq \mathcal{P}$. Then*

$$ht_{\frac{R}{I}}\left(\frac{\mathcal{P}}{I}\right) \le ht_R(\mathcal{P}) \le ht_{\frac{R}{I}}\left(\frac{\mathcal{P}}{I}\right) + n.$$

Proof. By Lemma 1.8 we have $I_{\mathcal{P}}$ is a proper ideal of Noetherian ring $R_{\mathcal{P}}$, and it can be generated by n elements. Hence using Corollary 1.21 we get that

$$ht_{\frac{R_{\mathcal{P}}}{I_{\mathcal{P}}}}\left(\frac{\mathcal{P}_{\mathcal{P}}}{I_{\mathcal{P}}}\right) \le ht_{R_{\mathcal{P}}}(\mathcal{P}_{\mathcal{P}}) \le ht_{\frac{R_{\mathcal{P}}}{I_{\mathcal{P}}}}\left(\frac{\mathcal{P}_{\mathcal{P}}}{I_{\mathcal{P}}}\right) + n.$$

and then by Corollary 2.3 we get that

$$ht_{\frac{R}{I}}\left(\frac{\mathcal{P}}{I}\right) \le ht_R(\mathcal{P}) \le ht_{\frac{R}{I}}\left(\frac{\mathcal{P}}{I}\right) + n.$$

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