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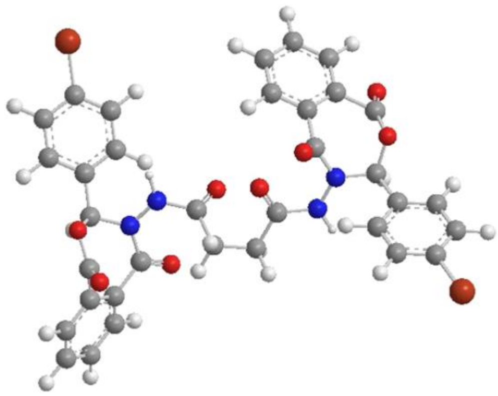

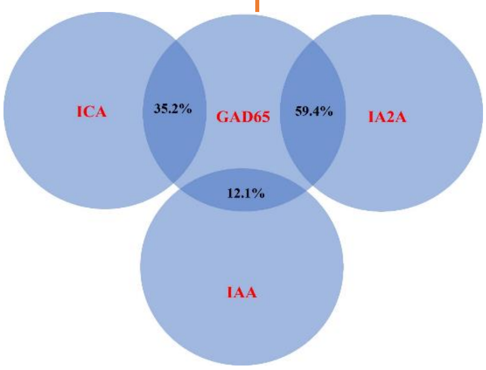

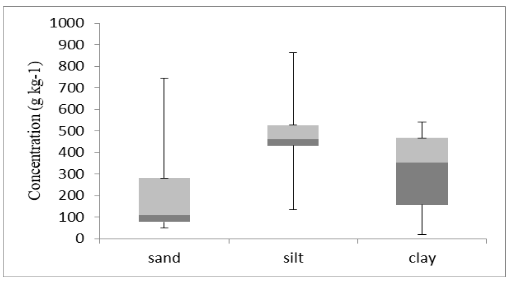

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Iterative method improving Newton's method with higher order

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Abstract

In the present article, we have constructed and looked at improving two- and three-step iterative methods to locate simple roots of non-linear equations. It appears that the three-step iterative method converges to the eighth order. The whole aim of this research is to derive and present a new modified iterative method of higher order and to obtain less iteration than the classical Newton method for solving nonlinear equations of simple roots. The present technique has to evaluate three functions and two first derivatives in each iteration. The advantage of the proposed method has been observed to have at least better performance effective and more stability by comparing with the other methods for the same or less than order. Also, noted that our method gives better results in terms of the number of iterations. To demonstrate the efficacy and popularity, different numerical illustrations are provided for the recommended techniques.

Introduction

One of the most important and challenging problem in scientific computing is to find the solutions of nonlinear equations (1). There exists a large number of applications such as chemical engineering transportation, operation research that give rise to thousands of such equations depending on one or more parameters (9). In this article, we consider an iterative method for finding the root of nonlinear equations $g(x)=0$. A nonlinear equation can be approximated using a number of methods, such as the Newton-Raphson method with quadratic convergence (2, 3). Multipoint iterative methods for finding solutions of non-linear equations have been a constant interesting field of study in numerical analysis. Important research on these methods was carried out during the last ten years. Several authors (3, 5, 6, 7, 13) attempted to develop higher order methods by adding finite evaluations of function in the multipoint methods to obtain less iteration than the classical Newton method. In order to solve the nonlinear equation, Abbasbandy (1) recently modified the Newton-Raphson method, using the Adomain decomposition method. Chun (4) has developed and evaluated a variety of higher order convergent one-step and two-step iterative methods. Noor and Noor both thought of a different decomposition method without a derivative. Kou and Li (9) developed a family of fifth order methods. Also another decomposition method has been taken into consideration by Noor and Noor (5). Siyyam (6) developed and tested a new fourth-step iterative technique with a fifth order of convergence, as well as additional improvements and changes to current methods and composite iterative methods, see for example (6-12) .

Newton's method is the most popular approximate method for determining the nonlinear equation $g(x)=0$ which is defined by

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} \qquad x = 0,1,.... \tag{1}$$

This is important and basic methods, which converge quadratically (13) .

The classical Chebyshev-Halley method for simple roots (14), which improves Newton's method, has been described by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2}\frac{\tilde{p}_{g(x_n)}}{(1-\theta\,\tilde{p}_{g(x_n)})}\right)\frac{g(x_n)}{g'(x_n)} \tag{2}$$

Where

$$\tilde{p}_{g(x_n)} = \frac{g''(x_n)\,g(x_n)}{g'(x_n)^2}$$

Jarratt (15) constructed an optimal method described as having fourth-order convergence

$$w_n = x_n - \frac{2}{3}\frac{g(x_n)}{g'(x_n)}$$

$$x_{n+1} = x_n - k_{g(x_n)}\frac{g(x_n)}{g'(x_n)} \tag{3}$$

Where

$$k_{g(x_n)} = \left[1 - \frac{3}{2}\;\frac{g'(w_n) - g'(x_n)}{3g'(w_n) - 2g'(x_n)}\right]$$

Ostrowski’s method (2) with fourth-order convergence is given by

$$\begin{aligned} w_n &= x_n - \frac{g(x_n)}{g'(x_n)} \\ x_{n+1} &= x_n - \left(\frac{g(x_n)}{g(x_n) - 2g(w_n)}\right)\frac{g(w_n)}{g'(x_n)} \end{aligned} \tag{4}$$

In this article, we provide a new three-step technique by using suitable and appropriate combinations of the known methods given in introduction eq.(2).

Iterative Algorithms

The Newton's method could be modified to solve problems with third order convergence $g(x) = 0$, due to Zhou (7) that may be rewritten as the iterative scheme

$$x_{n+1} = x_n - \frac{g^2(x_n) - 2g(x_n)g(w_n)}{g(x_n)g'(x_n) - 3g'(x_n)g(w_n)} \tag{5}$$

This converges cubically in the neighborhood of α as well as the third order method (16) described by

$$x_{n+1} = x_n - \frac{g^2(x_n)}{g'(x_n)(g(x_n) - g(w_n))} \tag{6}$$

Where

$$w_n = x_n - \frac{g(x_n)}{g'(x_n)}$$

Suggest the linear combination (9) of formulas eq. (5) and eq. (6), which results in a class of higher order iterative methods and may be used to develop a new iterative method as below:

$$x_{n+1} = x_n - \partial \frac{g^2(x_n) - 2g(x_n)g(w_n)}{g(x_n)g'(x_n) - 3g'(x_n)g(w_n)} - (1-\partial)\frac{g^2(x_n)}{g'(x_n)(g(x_n) - g(w_n))} \tag{7}$$

Where $\partial \in R$ and $w_n = x_n - \frac{g(x_n)}{g'(x_n)}$

This is the called two-step iterative method of solving nonlinear equations.

Obviously, for $\partial = 0$, formula eq. (7) reduce to eq. (6), while we have eq. (5) for $\partial = 1$.

However, to derive a new three-step iterative method combine eq. (7) and Newton's method eq. (1), As a result, the new method's local order of convergence improves from four for the eq. (7) method to at least eight.

$$\begin{aligned} w_n &= x_n - \frac{g(x_n)}{g'(x_n)} \\ v_n &= x_n - \partial \frac{g^2(x_n) - 2g(x_n)g(w_n)}{g(x_n)g'(x_n) - 3g'(x_n)g(w_n)} - (1-\partial)\frac{g^2(x_n)}{g'(x_n)(g(x_n) - g(w_n))} \\ x_{n+1} &= v_n - \frac{g(v_n)}{g'(v_n)} \end{aligned} \tag{8}$$

To find the order of convergence of eq. (8) and discuss about the choices of the parameter ∂ , we prove the following theorem:

Convergence Analysis:

This section looks at the new introduced methods' convergence analysis eq. (7) and eq. (8).

Theorem 1:

Assume that the function $g: X \subset \Re \to \Re$ for an open interval X has a simple root $\alpha \in X$, If $g(x)$ is a sufficiently smooth function close by of the root α , then the iterative method defined in equation (7) has at least fourth order convergence if $\partial = \frac{1}{2}$.

Proof:

Let $e_n = x_n - \alpha$, n=1,2,…

By using Taylor expansions about α , we can write

$$g(x_n) = g'(\alpha)[\, e_n + b_2 e_n^2 + b_3 e_n^3 + b_4 e_n^4 + O(e_n^4)] \tag{9}$$

$$g(\alpha) = 0,$$

$$g'(x_n) = g'(\alpha)[\,1 + 2b_2 e_n + 3b_3 e_n^2 + 4b_4 e_n^3 + 5b_5 e_n^4 + O(e_n^4)] \tag{10}$$

Where

$$b_j = \frac{g^{(j)}(\alpha)}{j!\; g'(\alpha)}, \quad \text{for} \quad j = 2, 3, \ldots$$

By dividing eq. (9) by eq. (10), we get

$$\frac{g(x_n)}{g'(x_n)} = e_n - b_2 e_n^2 + 2(b_2^2 - b_3) e_n^3 + O(e_n^4). \tag{11}$$

From eq. (11), we get

$$w_n = x_n - \frac{g(x_n)}{g'(x_n)} - \alpha = b_2 e_n^2 + 2(b_2^2 - b_3)\, e_n^3 + O(e_n^4). \tag{12}$$

Also, by using Taylor expansion about α , we have

$$g(w_n) = b_2\; e_n^2 + 2\,(b_2^2 - b_3)\, e_n^3 + O\,(e_n^4). \tag{13}$$

As a result, when eq. (9), eq. (10) and eq. (13) are substituted into eq. (7), we obtain the following error:

$$e_{n+1} = (-2\,\partial\, b_2^2 + b_2^2)\;\; e_n^3 + O(e_n^4). \tag{14}$$

If $\partial = \frac{1}{2}$ we have the error equation as

$$e_{n+1} = -\,(b_2\; b_3)\, e_n^4 + O(e_n^5). \tag{15}$$

This shows that the iteration methods given in equation (7) have the fourth order convergence.

Its proved.

The following method is obtained by putting $\partial = \frac{1}{2}$ in equation (7).

Algorithm 1:

Step 1 Let n=0 , to start with x_0 calculate x_1, x_2 ,… such that

$$w_n = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$x_{n+1} = x_n - \frac{1}{2}\frac{g^2(x_n) - 2g(x_n)g(w_n)}{g(x_n)g'(x_n) - 3g'(x_n)g(w_n)} - \frac{1}{2}\frac{g^2(x_n)}{g'(x_n)(g(x_n) - g(w_n))}$$

Step 2 for a given $\varepsilon > 0$, if $|f(x_n)| < \varepsilon$, then stop.

Step3 set n+1 then go to step one.

Theorem 2:

Assume that the function $g : X \subset \Re \to \Re$ for an open interval X has a simple root $\alpha \in X$, If $g(x)$ is a sufficiently smooth function close by of the root α , then the iterative method defined in equation (8) has at least eighth order convergence if $\partial = \frac{1}{2}$.

Proof:

Since from eq. (15) we have

$$v_n = e_n - (-2\partial\, b_2^2 + b_2^2)\ e_n^3 + O(e_n^4). \tag{16}$$

Again extending $g(v_n)$ about α we obtain

$$g(v_n) = g'(\alpha)[(-3\partial\, b_2^2 + 2b_2^2)\ e_n^3 + O(e_n^4)] \tag{17}$$

And $g'(v_n) = g'(\alpha)[1 + 2(-3\partial b_2^2 + 2b_2^2)b_2\ e_n^3 + O(e_n^4)]$ (18)

As a result, when eq. (16), eq. (17) and eq. (18) are substituted into eq. (8), the error equation is obtained.

$$e_{n+1} = (b_2^5(2\partial - 1)^2)\ e_n^6 + O(e_n^7)\,. \tag{19}$$

Furthermore, from eq. (19), when $\partial = \frac{1}{2}$, the method has at least eighth order of convergence and we have the error equation as

$$e_{n+1} = (b_2^3\, b_3^2)e_n^8 + O(e_n^9). \tag{20}$$

This ends the proof.

By putting $\partial = \frac{1}{2}$ in eq. (8), we obtain the following algorithm with eighth order convergence.

Algorithm 2:

Step 1 let n=0, to start with x_0 calculate x_1, x_2 ,… such that

$$w_n = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$v_n = x_n - \frac{1}{2}\frac{g^2(x_n) - 2g(x_n)g(w_n)}{g(x_n)g'(x_n) - 3g'(x_n)g(w_n)} - \frac{1}{2}\frac{g^2(x_n)}{g'(x_n)(g(x_n) - g(w_n))}$$

$$x_{n+1} = v_n - \frac{g(v_n)}{g'(v_n)}$$

Step2 for a given $\varepsilon > 0$, if $|f(x_n)| < \varepsilon$, then pause.

Step3 set n+1 then go to step one.

It is clear that each iteration of the technique described by eq. (8) require three function evaluations. The efficiency index is defined by Gautschi (17) as $p^{\frac{1}{\Delta}}$, where p is the method's order and Δ is the number of function evaluations required for each iteration. We are aware that the techniques stated in eq. (7) have an efficiency index equal to $\sqrt[3]{4} \approx 1.587$ and the techniques described in eq. (8) have an efficiency index equal to $\sqrt[5]{8} \approx 1.5157$, both of which are superior to Newton's approach which equal to $\sqrt{2} \approx 1.414$.

Numerical example

We now provide a number of examples to show the effectiveness of the new techniques. We contrast Newton's methods (namely NM) eq. (1), Jarrett's method (namely JM) eq. (3), the method of Zhou (namely ZM) eq. (5), the method of Sharma (SHM) eq. (6) with the methods proposed in this paper Algorithm1 (namely AL1), AL2 and Algorithm2 (AL2). The first table displays the order of convergence for many methods. Table 2 shows the number of iterations (IT). For computer programs, we employ the following stopping criteria: $i)\ |g(x_{n+1})| < 1.E-32$ and $ii)\ |x_{n+1} - x_n| < 1.E-32$ found up to the 128th decimal place. The computations are performed using the software Maple 18.

In numerical comparisons, the following functions are used for the comparison and display the approximate zeros found up to the 20th decimal point.

$g_1(x) = \sin(x)^2 - x^2 + 1$ α =1.4044916482153412260350868891

$g_2(x) = \cos(x) - x$ α =0.7390851332151606486379400089

$g_3(x) = x^3 + 4x^2 - 10$ α =1.3652300134140968458239881 59

$g_4(x) = x^3 - 10$ α = 2.1544346900318837217592936 44

$g_5(x) = (x-1)^3 - 1$ α =2.0

$g_6(x) = e^{-x} + 2\sin x - x + 3.5$ α =3.2739381231367601554281023 80

$g_7(x) = e^{-x^2+x+2} - t + 2$ α =2.3314603605886482550027619 18

$g_8(x) = e^x - 1$ α =0.0

$g_9(x) = \tan^{-1} x + \ln(x^2+1) - 2$ α =1.3723245602103073765499736 48

$g_{10}(x) = x^5 + x^4 + 4x^2 - 15$ α =1.3474280989683049815067162 04

Table 1: shows the order in which various iterative methods converge.

	NM	ZM	SHM	JM	AL1	AL2
Order	2	3	3	4	4	8

Table 2: A comparison of the IT of several iterative techniques

$g(x)$	α	Number of iteration (IT)					
		NM	ZM	SHM	JM	AL1	AL2
g_1	1.0	7	4	5	4	3	2
	0.2	13	7	16	7	9	4
g_2	3.2	14	6	5	5	4	3
	6.0	11	5	4	4	4	3
g_3	2.0	6	4	4	3	3	2
	0.1	11	50	8	6	5	3
g_4	2.5	6	4	4	3	3	2
	6.0	8	14	5	4	4	3
g_5	-2.0	12	6	8	7	6	4
	0.0	10	13	7	10	5	4
g_6	1.6	36	5	5	4	4	3
	2.0	6	4	4	4	4	2
g_7	-2.25	7	5	5	7	5	3
	3.0	7	4	5	3	4	3
g_8	1.6	8	Div	5	4	5	4
	-2.5	16	Div	Div	6	5	3
g_9	8.0	5	5	3	3	3	3
	1.0	5	4	4	3	3	2
g_{10}	1.0	7	4	5	3	3	2
	2.5	8	Div	5	4	5	4

NOTE: Div strands for divergent.

Table 1 show that the algorithm AL1 obtained using the proposed Newton's method by two orders and the algorithm AL2 increase by four orders than the algorithm AL1. Also, from Table 2, Depending on the number of iterations, we could assume that the new methods suggested in this study employed better than the methods we evaluated. Furthermore, considering the numerical results, the computational results show that the current method takes less IT than NM, ZM, JM, and SHM, which means that the new method AL2 converges and faster than the other methods. Also, we see that the Algorithm2 is better than the Algorithm1. Consequently, the new method has useful applications.

Conclusions

We developed a new two- and three-step iterative method. The new methods increase the local order from three to four and eight, respectively. The methods give better results in terms of the number of iterations. Numerical tests also support higher-order convergence.

Conflict of interest

The authors confirm that they are not affiliated with or involved in any organization or entity with financial interests.

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