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Dynamics of an Eco-Epidemiological Model Consisting of Herding Prey and Harvested Predator

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Abstract

In this paper, the dynamics of predator-prey interaction, only at the outer surface of herd formed by prey population and spreading SI-disease in predator population, is modeled mathematically. The boundedness and persistence of our model are studied and local dynamical behaviors are investigated. Furthermore, the conditions that guarantee the occurring of Hopf-bifurcation of the model are established. Finally, we confirm our analytical findings with the help of numerical simulation

Introduction

The dynamics of interaction between species is one of the interesting subjects in mathematical modeling. The classical Lotka and Volterra model [6, 10] is the first major work for studying dynamics of prey-predator interaction. The mass action predation in the Lotka-Volterra model modified by new functional response [3-5]. M. Rosenzweig and R. MacArthur [9] have introduced prey predator with Holling type functional responses.

In last decades many researchers have modified the Lotka-Volterra model incorporating spreading disease on the species, prey refuge and harvesting factor [7-9,10]. In nature, for searching for food resource, defending the predator, many species forming herds and all members of the species do not interact at a time [1]. Braza [2], has placed strong emphasis on this concept and he replaced the prey density in Holling type-II function responses by its square root because the square root would count the individuals at the edge of the path. However, taking the square root is suitable for interaction at the edge of the path, if the prey density $X(t) \geq 1$, because if $X(t) < 1$, then$\sqrt{X(t)} > X(t)$ which means the interaction at the edge of the path of herding prey popualtion is more than the interaction with whole prey population which is impossible. Therfore, in this paper, we modified the model considered by Braza [2]. In our model the density X of herd prey in Holling type-II is replaced as follows:

$.\sqrt{X} \;\; if \; X \geq 1 \;\;\; and \; by \;\; X^2 \;\;\; if \;\;\; X < 1$

And we incorporate our model an epidemic disease factor in harvested predator. The paper is organized as follows. In the section 2, we have explained all the assumption in the proposed model. The positiveboundedness of all solutions of the model was introduced in section 3. Stability criterion for stability have been carried out and the conditions to guarantee the occurrence of Hopf-bifurcation of the model are established. Finally, we confirm our analytical findings with the help of numerical simulation in section 4.

2. The model Formulation:

The model we consider here is the following set of nonlinear differential equations

$$\frac{dx}{dt} = rX\left(1 - \frac{X}{k}\right) - f(X)(Y + Z)$$

$$\frac{dY}{dt} = ef(X)Y - \frac{\lambda YZ}{1+Z} - (d + h_1)Y$$

(1)

$$\frac{dZ}{dt} = \frac{\lambda YZ}{1 + Z} - (b + h_2)Z$$

where

and the initial values are $X_0 > 0, Y_0 > 0$ and $Z_0 > 0$.$f(x) = \begin{cases} \frac{\alpha\sqrt{X}}{1+\alpha T\sqrt{x}} & if\ X \geq 1 \\ \frac{\alpha X^2}{1+\alpha TX^2} & if\ \ X < 1 \end{cases}$

The model formulated on the following assumptions.

1. $X(t)$, $Y(t)$ and $Z(t)$ denote the density of prey, susceptible predator and infected predator respectively.
2. X(t) increasing due intrinsic growth with rate $r > 0$ and carring capacity $k > 0$.
3. The preyspecies live in herd and they predated by both (Susceptible as well as infected) predator according to the functional responses
$$f(x) = \begin{cases} \dfrac{\alpha\sqrt{X}}{1 + \alpha T\sqrt{x}} & if\ X \geq 1 \\ \dfrac{\alpha X^2}{1 + \alpha TX^2} & if\ \ X < 1 \end{cases}$$
where α the search efficiency of the predator for prey.T is the handling time for each prey
4. The infected prey donot reproduce therefore only the rate of biomass conversionof prey population to Susceptible predator isgiven and denoted by e and Susceptible predator and infected predator die out naturally with rate $d > 0$ and $b > 0$ respectivly.
5. It is assumed that the disease transmitted from the infected predator to Susceptible predator by contact according to the nonlinear incidence rate of the form $\frac{\lambda YZ}{1+Z}$[7].
6. susceptible predator and infected predator is harvested by the rate h_1 and $h_2$respectively.

3. Boundedness and Persistence:

It is easy to show that all solution of the system (1) with positive initial value remain positive and the uniformly bounded of all solution is shown in the theorem (1), and the condition, which guarantee the persistence of the system(1)is given in theorem(2).

Theorem 1. All solution of the system (1) with positive initial value are uniformly bounded.

Proof. From the first equation of the system(1), we have

So $\lim_{t\to\infty} \sup(X(t)) \leq k \frac{dx}{dt} \leq rX\left(1-\frac{X}{k}\right)$

From the system (1), we get

$$\frac{d(X+Y+Z)}{dt} \leq (r+\beta)X + (e-1)f(x) - \beta(X+Y+Z)\text{ , where } \beta = min\{d+h_1, b+h_2\}$$

Now, from the biological point of view, always $e \leq 1$, because the conversation rate from prey to susceptible predator can not be exceeding the maximum predation rate of susceptible predator to prey. Hence it is obtained that

$$\frac{d(X+Y+Z)}{dt} \leq (r+\beta)X - \beta(X+Y+Z) \leq (r+\beta)k - \beta(X+Y+Z)$$

So, $\lim_{t\to\infty}(X(t)+Y(t)+Z(t) \leq \frac{(r+\beta)k}{\beta}$. Thus, all solution of system (1) enter in to the bounded region

$$R = \left\{(X,Y,Z): 0 < X+Y+Z \leq \frac{(r+\beta)k}{\beta}\right\}$$

Theorem 2. For the initial value $X_0 > 0$, $y_0 \geq \frac{(\lambda+d+h_1)\frac{(r+\beta)k}{\beta}}{ef(\sigma)}$ and $Z_0 \geq \frac{ef(\sigma)(b+h_2)\left(1+\frac{(r+\beta)k}{\beta}\right)}{\lambda(\lambda+d+h_1)}$, the system(1) is persist if the following condition holds

$$(2) k(r+\beta) < rT\beta$$

Proof. From first equation of the system(1), we have

$$\frac{dx}{dt} \geq rX\left(1-\frac{X}{k}\right) - \frac{1}{T}(Y+Z)$$

From Theorem (1). It is obtained that $\lim_{t\to\infty} Sup(X(t)+Y(t)+Z(t)) \leq \frac{(r+\beta)k}{\beta}$,

thus $\frac{dx}{dt} \geq rX\left(1-\frac{(r+\beta)k}{rT\beta}-\frac{X}{k}\right)$ as time approaches infinity.

Solving the above differential inequality we get, $\lim_{t\to\infty} \inf(X(t)) \geq \left(1-\frac{(r+\beta)k}{rT\beta}\right)k$

Now due to condition(2), we have $\left(1-\frac{(r+\beta)k}{rT\beta}\right)k > 0$

Let $\left(1-\frac{\alpha(r+\beta)k}{r\beta}\right)k=\sigma$, then from second equation of the system(1). It is obtained that

as time approaches infinity, again solving this differential $\frac{dY}{dt}\geq ef(\sigma)Y-(\lambda+d+h_1)\frac{(r+\beta)k}{\beta}$ inequality we get,

. If $y_0\geq\frac{(\lambda+d+h_1)\frac{(r+\beta)k}{\beta}}{ef(\sigma)}\lim_{t\to\infty}\inf\big(Y(t)\big)\geq\frac{(\lambda+d+h_1)\frac{(r+\beta)k}{\beta}}{ef(\sigma)}>0$

Now from the third equation of the model(1), we get,

$$\frac{dZ}{dt}\geq\frac{\lambda(\lambda+d+h_1)\frac{(r+\beta)k}{\beta}}{ef(\sigma)\left(1+\frac{(r+\beta)k}{\beta}\right)}Z-(b+h_2)\frac{(r+\beta)k}{\beta}$$

and hence

if $Z_0\geq\frac{ef(\sigma)(b+h_2)\left(1+\frac{(r+\beta)k}{\beta}\right)}{\lambda(\lambda+d+h_1)}\lim_{t\to\infty}\inf\big(Z(t)\big)\geq\frac{ef(\sigma)(b+h_2)\left(1+\frac{(r+\beta)k}{\beta}\right)}{\lambda(\lambda+d+h_1)}>0$

4. Stability analysis and bifurcation :

Before discussing the stability analysis of each equilibrium points, we have the following theorem on the extinction of both Susceptible and infected predator .

Theorem 3. If the following condition holds then $\lim_{t\to\infty}(Y+Z)=0$.

$$\theta\leq\frac{e}{T} \quad (3)$$

Where $\theta=min\{(d+h_1),\ (b+h_2)\}$

Proof. From the second and the third equations of the system(1), we get

$$\frac{d(Y+Z)}{dt}=ef(X)Y-(d+h_1)Y-(b+h_2)Z$$

So, $\frac{d(Y+Z)}{dt}\leq\frac{e}{T}Y-(d+h_1)Y-(b+h_2)Z\leq\frac{e}{T}(Y+Z)-\theta(Y+Z)=\left(\frac{e}{T}-\theta\right)(Y+Z)$

Hence due condition(3), we have $\lim_{t\to\infty}(Y+Z)=0$.

4.1 The trivial equilibrium:

The trivial equilibrium point $E_1=(0,0,0)$ is always exist and variational matrix at E_1 is written as

$$V(E_1)=\begin{pmatrix} r & 0 & 0\\ 0 & -(d+h_1) & 0\\ 0 & 0 & -(b+h_2)\end{pmatrix}$$

So, one of eigenvalues of $V(E_1)$ is always positive which guaranties that $E_1=(0,0,0)$ is unstable always, and hence there is no possibility to have a Hopf- bifurcation near this point.

4.2. The axial equilibrium:

The axial equilibrium point $E_2 = (k, 0,0)$ is always exist and variational matrix at E_1 is written as

$$V(E_2) = \begin{pmatrix} -r & -f(k) & -f(k) \\ 0 & ef(k) - (d + h_1) & 0 \\ 0 & 0 & -(b + h_2) \end{pmatrix}$$

Clearly the two eigenvalues $-r \; and \; -(b + h_2)$, are always negative so, there is no possibility to have a Hopf bifurcation near this $E_2 = (k, 0,0)$ .and the eigenvalue $ef(k) - (d + h_1)$ is negative if the condition(4) holds,

$$(4) d + h_1 > ef(k)$$

and hence$E_2 = (k, 0,0)$ is locally asymptotically stable.

4.3 The infected predator free equilibrium:

The density values infected predator free equilibrium $E_3 = (\tilde{X},\tilde{Y}, 0)$are

$$\tilde{X} = \left(\frac{d + h_1}{\alpha(\mathrm{e} - \mathrm{T}(\mathrm{d} + \mathrm{h}_1))}\right)^2 \quad and \; \tilde{Y} = \frac{er}{d + h_1}\tilde{X}\left(1 - \frac{\tilde{X}}{k}\right)$$

If the following condition (6)hold

$$(6) \left.\begin{matrix} d + h_1 > \alpha(\mathrm{e} - \mathrm{T}(\mathrm{d} + \mathrm{h}_1)) > 0 \\ and \quad \left(\frac{d + h_1}{\alpha(\mathrm{e} - \mathrm{T}(\mathrm{d} + \mathrm{h}_1))}\right)^2 < k \end{matrix}\right]$$

But $\tilde{X} = \sqrt{\frac{d + h_1}{\alpha(\mathrm{e} - \mathrm{T}(\mathrm{d} + \mathrm{h}_1))}} \quad and \; \tilde{Y} = \frac{er}{d + h_1}\tilde{X}\left(1 - \frac{\tilde{X}}{k}\right)$

If the following condition (7)hold

$$(7) and \left.\begin{matrix} d + h_1 < \alpha(\mathrm{e} - \mathrm{T}(\mathrm{d} + \mathrm{h}_1)) \\ \sqrt{\frac{d + h_1}{\alpha(\mathrm{e} - \mathrm{T}(\mathrm{d} + \mathrm{h}_1))}} < k \end{matrix}\right]$$

At the infected free predator equilibrium point the variational matrix is

$$V(E_3) = \begin{pmatrix} r - 2r\frac{\tilde{X}}{k} - \frac{df}{dx}(\tilde{X})\tilde{Y} & -f(\tilde{x}) & -f(\tilde{X}) \\ e\frac{df}{dx}(\tilde{X})\tilde{Y} & 0 & -\lambda\tilde{Y} \\ 0 & 0 & \lambda\tilde{Y} - (b + h_2) \end{pmatrix}$$

So, all the eigenvalues of $V(E_3)$have negative real part if the condition (8) holds, and hence $E_3 = (\tilde{X}, \tilde{Y}, 0)$locally asymptotically stable.

$$(8)\frac{r-2r\frac{\alpha\tilde{X}}{k}}{\frac{df}{dx}(\tilde{X})} < \tilde{Y} < \frac{(b+h_2)}{\lambda}$$

The condition on parameters for occurrence of Hopf bifurcation near the infected predator free equilibrium point isgiven in theorem(4).

Theorem 4 . In addition condition (6) if the following conditions(9)and (10)hold , then system (1) has a Hopf bifurcation near the infected predator free equilibrium point as the parameter value k passes through the value $k = \bar{k}$, where $\bar{k}$ is given in the proof.

$$(9)\tilde{Y} < \frac{(b+h_2)}{\lambda}$$

$$(10)\frac{e\alpha\bar{X}}{2(d+h_1)\sqrt{\tilde{X}}(1+\alpha T\tilde{X})^2} < 1$$

Proof. According to the vibrational matrix of the system (1) at the infected predator free equilibrium point$E_3 = (\tilde{X}, \tilde{Y}, 0)$, it is easy to verify that the its characteristic equation can be written as follows

$$\left(\mu - \lambda\tilde{Y} + (b + h_2)\right)(\mu^2 + A_1\mu + A_2) = 0$$

Where $A_1 = 2r\frac{\tilde{X}}{k} + \frac{er}{d+h_1}\tilde{X}\left(1 - \frac{\tilde{X}}{k}\right)\frac{df}{dx}(\tilde{X}) - r$ and $A_2 = ef(\tilde{x})\frac{df}{dx}(\tilde{X}) > 0$

Thus the eigenvalue in Z –direction is $\lambda\tilde{Y} - (b + h_2)$ which is negative due condition(9)and the eigenvalues in X –direction and Y –direction are $\frac{-A_1}{2} \pm \frac{1}{2}\sqrt{A_1^2 - 4A_2}$

Clearly, as shown above,both of the eigenvalues in XY-plane are pure imaginary complex numbers for $A_1 = 0$. Now due to condition(10) at the value $k = \frac{\bar{X}\left(2 - \frac{e\alpha\bar{X}}{2(d+h_1)\sqrt{\tilde{X}}(1+\alpha T\tilde{X})^2}\right)}{\left(1 - \frac{e\alpha\bar{X}}{2(d+h_1)\sqrt{\tilde{X}}(1+\alpha T\tilde{X})^2}\right)} = \bar{k}$which is positive,$A_1 = 0$,so, there is a neighborhood around $k = \bar{k}$such that both eigenvalues in XY-plane be written as

$$w(\bar{k}) \pm iw(\bar{k})$$

Where $w(k) = 2r\frac{\alpha\tilde{X}}{k} + \frac{er}{d+h_1}\tilde{X}\left(1-\frac{\tilde{X}}{k}\right)\frac{df}{dx}(\tilde{X}) - r.$

Now, since $\frac{dw(k)}{dk} = -r\frac{\alpha\tilde{X}}{k^2} + \frac{er\tilde{X}^2}{k^2(d+h_1)}\frac{df}{dx}(\tilde{X}) \neq 0$

Therefore, system (1) has a Hopf bifurcation near the infected predator free equilibrium point at

and hence the proof is complete.$k = \bar{k}$

4.4 The positive equilibrium point:

The positive equilibrium$E_4 = (\hat{X}, \hat{Y}, \hat{Z})$, is solution for the following system

$$rX\left(1-\frac{X}{k}\right) = f(X)(Y+Z)$$
$$ef(X) = \frac{\lambda Z}{1+Z} + (d+h_1)$$
$$\frac{\lambda Y}{1+Z} = (b+h_2)$$

Thus,

, $\hat{Z} = \frac{ef(X)-(d+h_1)}{\lambda-ef(X)+(d+h_1)}$ and $\hat{X}$ is solution of the following equation $\hat{Y} = \frac{1}{\lambda}(b+h_2)\left(\frac{\lambda}{\lambda-ef(X)+(d+h_1)}\right)$

$$X\left(1-\frac{X}{k}\right) = f(X)\left(\frac{b+h_2+ef(X)-(d+h_1)}{\lambda-ef(X)+(d+h_1)}\right)$$

The condition for locally stability of the equilibrium and accuracy of hopfbifurcation near it given in the following theorem

Theorem 5. Suppose that $E_4 = (\hat{X}, \hat{Y}, \hat{Z})$ exist then

i. $E_4 = (\hat{X}, \hat{Y}, \hat{Z})$ is locally asymptotically stable iff $D_1 > 0, D_3 > 0$ and $D_1 D_2 > D_3$, where D_1, D_2 and D_3 are given in the proof

ii. If there is a carrying capacity parameter $\bar{\bar{k}}$ at which $D_1 D_2 = D_3$ and $D_1 > 0$ then the system(1) exhibits a Hopfbifurcation near $E_4 = (\hat{X}, \hat{Y}, \hat{Z})$ as the carrying kcapacity passes through $\bar{\bar{k}}$.

Proof i. Straight forward computation shows that the variational matrix at near $E_4 = (\hat{X}, \hat{Y}, \hat{Z})$ is

$$V(E_4) = \begin{pmatrix} r - 2r\frac{\alpha\hat{X}}{k} - \frac{df}{dx}(\hat{X})(\hat{Y}+\hat{Z}) & -f(\hat{X}) & -f(\hat{X}) \\ e\frac{df}{dx}(\hat{X})\tilde{Y} & 0 & -\frac{\lambda\hat{Y}}{(1+\hat{Z})^2} \\ 0 & \frac{\lambda\hat{Z}}{(1+\hat{Z})^2} & \frac{\lambda\tilde{Y}}{(1+\hat{Z})^2} - (b+h_2) \end{pmatrix}$$

Then the characteristic equation of $V(E_4)$ can be written as follows

$$\mu^3 + D_1\mu^2 + D_2\mu + D_3 = 0$$

Where $D_1 = 2r\frac{\alpha\hat{X}}{k} + \frac{df}{dx}(\hat{X})(\hat{Y}+\hat{Z}) + (b+h_2) - \frac{\lambda\hat{Y}}{(1+\hat{Z})^2} - r$

$$D_2 = ef(\hat{X})\frac{df}{dx}(\hat{X})\tilde{Y} + \frac{\lambda^2\widehat{Y\hat{Z}}}{(1+\hat{Z})^4} + \left(r - 2r\frac{\alpha\hat{X}}{k} - \frac{df}{dx}(\hat{X})(\hat{Y}+\hat{Z})\right)\left(\frac{\lambda\tilde{Y}}{(1+\hat{Z})^2} - (b+h_2)\right)$$

$$D_3 = \left(\frac{e\lambda\hat{Y}(\hat{Y}-\hat{Z})}{(1+\hat{Z})^2} - e\hat{Y}(b+h_2)\right)f(\hat{X})\frac{df}{dx}(\hat{X}) - \frac{\lambda^2\hat{Z}\hat{Y}}{(1+\hat{Z})^4}\left(r - 2r\frac{\alpha\hat{X}}{k} - \frac{df}{dx}(\hat{X})(\hat{Y}+\hat{Z})\right)$$

So according to the Routh-Hurwitz criterion $E_4 = (\hat{X},\hat{Y},\hat{Z})$ is locally asymptotically stable iff the given conditions in the theorem holds.

Proof ii. Since at $\bar{\bar{k}}$ we have $D_1D_2 = D_3$ so the eigenvalues for the matrix $V(E_4)$at $\bar{\bar{k}}$ are

and $\mu_3 = -D_1$ and since $D_1 > 0$ and $D_2 > 0$ so, two of the eigenvalues are pure $\mu_{1,2} = \pm i\sqrt{D_2}$ imaginary complex numbers and the thirdeigenvalue is negative.

However, for all carrying capacity parameter k in the neighborhood of $\bar{\bar{k}}$, these eigenvalues can be written as $\mu_{1,2} = \epsilon(k) \pm i\delta(k)$ and $\mu_3 = -D_1(k)$ now puuting the complex in the charastriti euation and taking derivative with respect k, get

$$\frac{d}{dk}Re(\epsilon(k)) = \frac{A_1A_2 + A_3A_4}{A_1^2 + A_3^2}$$

Where

$$A_1 = 3(\epsilon(k))^2 + 2D_1(k)\epsilon(k) + D_2(k) - 3(\delta(k))^2$$

$$A_2 = (\epsilon(k))^2 D_1^{'}(k) + D_2^{'}(k)\epsilon(k) + D_3^{'}(k) - D_1^{'}(k)(\delta(k))^2$$

$$A_3 = 6\epsilon(k)\delta(k) + 2D_1(k)\delta(k)$$

$$A_4 = 2\epsilon(k)\delta(k)D_1^{'}(k) + D_2^{'}(k)\delta(k)$$

It is easy to verify that $\frac{A_1A_2+A_3A_4}{A_1^2+A_3^2} \neq 0$ at $k = \bar{\bar{k}}$ and $\mu_3 = -D_1(\bar{\bar{k}}) < 0$. Hence the proof is complete.

5. Numerical Simulation:

In this section, the dynamics of the system(1) is investigated numerically using Euler' Method with help of Matlab.

For the following set of biological feasible parameters,$\bar{\bar{k}} = 11.63$ and the system is solved numerically setting $k = 11 < \bar{\bar{k}}$ and the trajectory is drawn in the figure (1).

$r = 1.1, \alpha = 0.3, T = 0.3333, e = 0.99, \lambda = 0.5,$

(11)$and\ d = h_1 = b = h_2 = 0.1.$

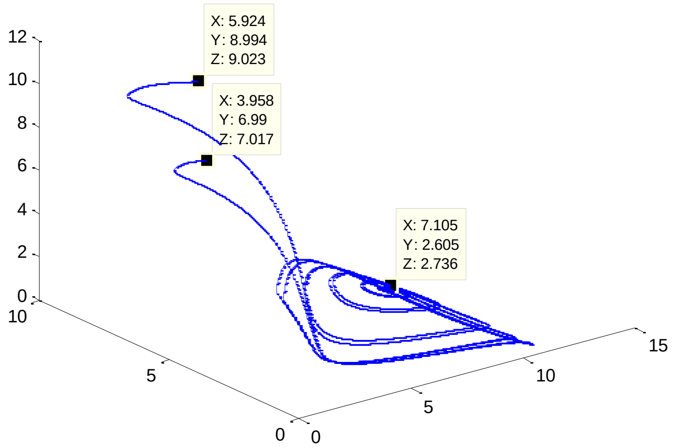


Fig. 1. Two different trajectories of system(1)) approaches asymptotically the positive equilibrium point (7.1, 2.6, 2.73).

According to above figure the system(1) is stabilized for all initial value near the equilibrium point (7.1, 2.6, 2.73) for the parameter given in (11) with $k = 11$ which less than $\bar{\bar{k}}$. Note that, the figure(1)confirms our analytical result for stability of positive equilibrium point because the parameters satisfies the stability condition..

Now again we solve the system by same date set of parameters in (11) with $k = 11.6$ and the trajectory plotted in the figure (2)

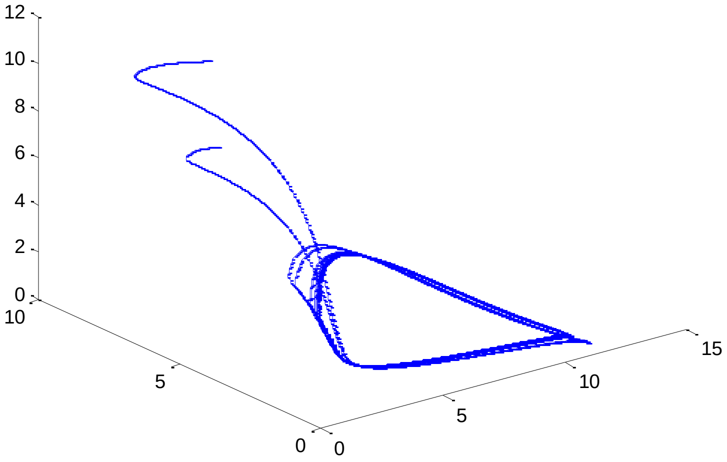


Fig (2). The system(1) approaches a periodic attractor for $k = 11.65$ with the rest of parameters given by equation (11) clearly, figure (2) shows that system losses its stability and approaches aperiodic attractor as the carrying capacity increases to $k = 11.65$ which is greater than $\bar{\bar{k}}$. Which means the system undergo a Hopf bifurcatin as the carrying capacity parameter passes through $\bar{\bar{k}} = 11.63$, which confirm our result in theorem(5).

Now to explain the effect of disease and harvesting on the system, solved the system numerically by for the parameters given in equation with decreasing the infective rate to $\lambda = 0.001$ and increases the harvesting rate on infected predator to $h_2 = 0.5$ and for both carryingcapacity less than $\bar{\bar{k}} = 11.63$as shown in figure (3) and greater than$\bar{\bar{k}} = 11.63$ as shown in figure (4).

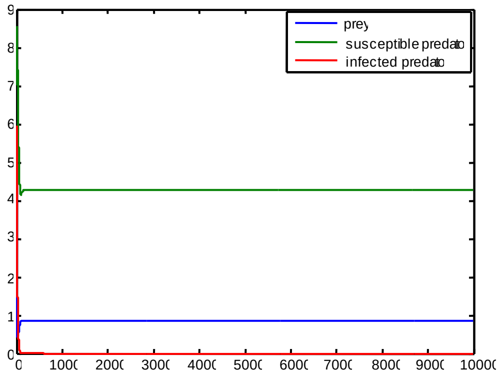


Fig (3) the systemapproaches the infected predator free equilibrium point for the parameters given by equation (11) with $k = 11, \lambda = 0.0001$ and $h_2 = 0.5$

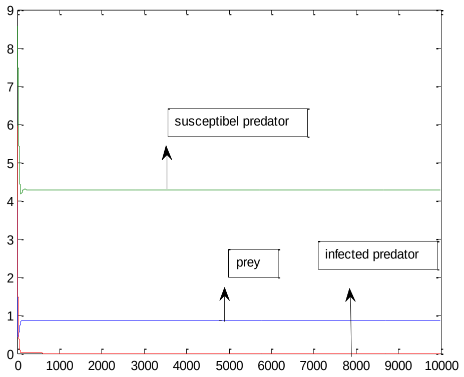


Fig (4) the systemapproaches the infected predator free equilibrium point for the parameters given by equation (11) with $k = 12, \lambda = 0.0001$ and $h_2 = 0.5$

Fig.(3) and fig.(4) shown that, the system for both the carrying capacity parameter values doesnot undergo hopf bifurcation if we decrease the infective rate to $\lambda = 0.001$ and increases the harvesting rate on infected predator to $h_2 = 0.5$. not that the parameters used in Fig (3) and Fig(4), satisfies the condtion which cinfirm our analytical result.

again if we solve the system (1) for the parameters given in equation (11) with where $k = 11.63$ is bifurcation parameter if we donot $k = 11.63, \lambda = 0.0001$ and $h_2 = 0.5, h_2 = 0.8$ change the parameters in equation, the system approaches asymptotically to the axial point as shown in Fig.(5).

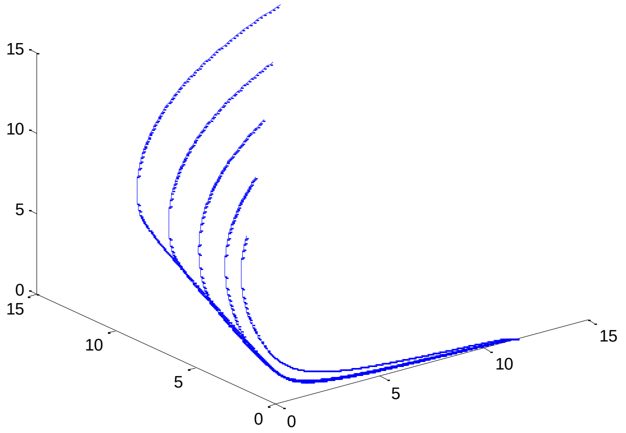


Fig. (5) the systemthat started at different initial point, approaches the axial equilibrium point$(11.63,0,0)$ for the parameters given by equation (11) with

$k = 11.63, \lambda = 0.0001$ and $h_2 = 0.5, h_2 = 0.8$

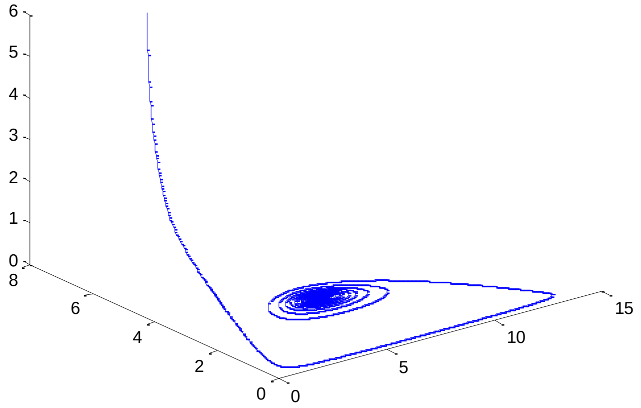
Fig.(5) shows that if the system (1) solved analytical with parameter satisfies the condition it approaches the axial equilibrium which confirm our analytical result for stability of axial point and the system does not have hopfbifurcation this is explained the role of infective disease and harvesting factor.

Finally the dynamics of the system(1) solve analytical for the following data set and different values of capacity it is observed that the system undergo hopfbifurcation as carrying capacity parameter passes through$\bar{k} \cong 14.5$ which confirm result in theorem (5) as shown in Fig.(6-8).

$r = 1.1, \alpha = 1, T = 0.5, e = 0.9, \lambda = 1,$

(12)and $d = .5, h_1 = 0.5, b = h_2 = 1.5.$

The system(1) approaches an infected free equilibrium for $k = 14$ with the rest of parameters in eq(12)



Fig(6)The system(1) approaches an infected free equilibrium for $k = 14$ with the rest of parameters given by equation (12)

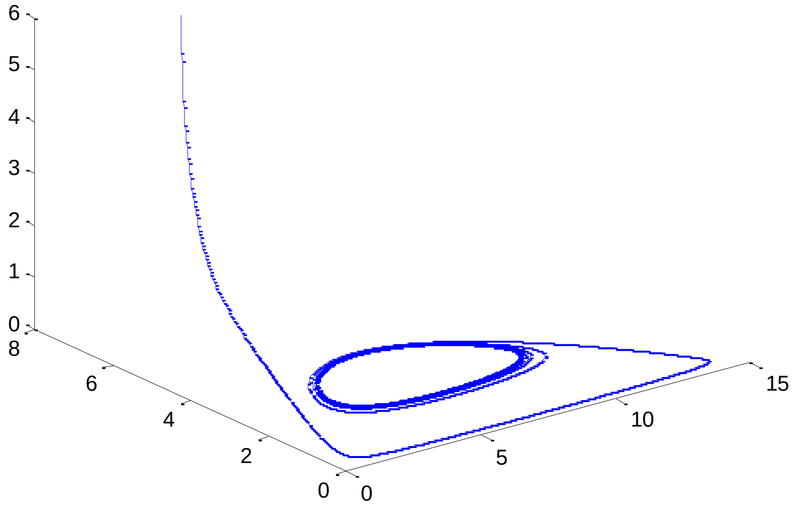
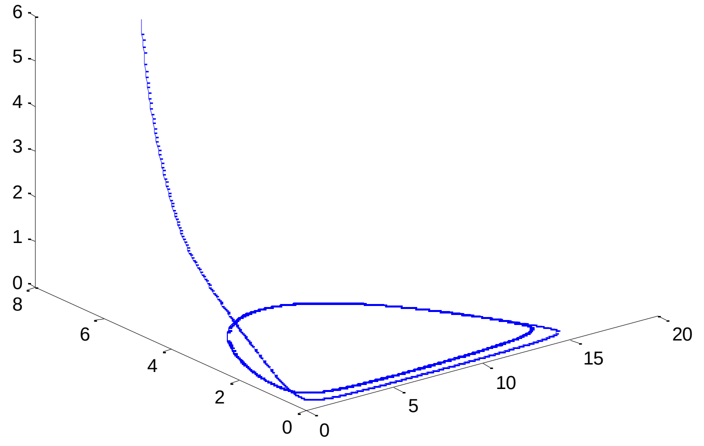


Fig.(7)The system(1) approaches a small periodic attractor for $k = 14.8$ with the rest of parameters given in Eq (12)



Fig(8)The system(1) approaches a large periodic attractor for $k = 11.65$ with the rest of parameters given by equation (12)

6. Conclusions:

In this work, we modified the model considered by Braza [2], with the following

modifications:

i. Modifying the functional responses, he replaced the prey density in Holing type-II function responses by its square root because the square root would count the individuals at the edge of the path. In our model the density X of herd prey in Holling type-II functional responses is replaced as follows:

$$.\sqrt{X} \quad if\ X \geq 1 \quad and\ by\ X^2 \quad if \quad X < 1$$

ii. Incorporating the model an epidemic disease and harvesting factor on predator species.

Then, boundedness, persistence, stability and Hopf bifurcation of system(1) are studied with confirming our results numerically. In our numerical simulation, we have found that for certain parameter values the system approaches a positive equilibrium or a periodic attractor which means the system is persistent. However, when we decrease the effect of the epidemic disease and increase the effect of harvesting factor, the system approaches the infected free equilibrium or axial equilibrium which means the infected predator or whole predator will disappear.

7. References

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