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On Generalized Flatness and Generalized SF-Rings

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Abstract

The concept of generalized flatness and generalized SF-rings were first introduced by the author in 2007. Now, in the present paper we continue to study generalized flatness, generalized SF-rings and determine more properties of them. Moreover, several results are proved. Finally, compare them with regular rings, left non-singular, strongly regular, strongly π-regular, π-biregular and sπ-weakly regular rings under some conditions.

I. Introduction and Preliminaries

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. The Jacobson radical [7] of a ring R, denoted by J(R) is the intersection of all maximal ideals of R. For a subset X of R, the right annihilator of X in a ring R is defined by $r(X) = \{t \in R: xt = 0, \text{ for all } x \in X\}$. Similarly, define the left annihilator of X in a ring R as $\ell(X) = \{t \in R: tx = 0, \text{ for all } x \in X\}$. If X={a}, we usually use r(a) (ℓ(a)). An ideal I of a ring R is said to be essential if and only if I has a non-zero intersection with every non-zero ideal of R. Let x be an element in R. Then x is said to be right (left) singular if and only if r(x) (ℓ(x)) is essential right (left) ideal of R. The set of all right (left) singular ideals in R is denoted by Y(R) (Z(R)). Y(R) (Z(R)) is a right (left) ideal of R, which is called the right (left) singular ideals of R. R is called right (left) non-singular if Y(R) = (0) (Z(R) = (0)) and it is called semi-primitive [8] if J(R) = (0). R is called reduced [8] if it contains no non-zero nilpotent elements, or equivalently, $a^2 = 0$ implies a = 0, for all $a \in R$ and it is called a zero insertive (briefly, ZI) ring [11] if for any $a, b \in R$, ab = 0 implies that aRb = 0. R is called semi-prime [7] if it contains no non-zero nilpotent ideal. An ideal I of a ring R is said to be a nil ideal [7] if every element of I is a nilpotent element. R is called right (left) quasi-duo [30] if every maximal right (left) ideal of R is two-sided and it is called MERT-ring [6] if every maximal essential right ideal of R is two-sided. R is called von Neumann (or, just regular) [27] if for every $a \in R$, there exists $b \in R$ such that a = aba and it is called π-regular [14] if for every element $a \in R$, there exists $b \in R$ and a positive integer n such that $a^n = a^n b a^n$. R is said to be strongly regular [8] if for every $a \in R$, there exists $b \in R$ such that $a = a^2 b$ and it is called strongly π-regular [4] if for every $a \in R$, there exists a positive integer n and $x \in R$ such that $a^n = a^{n+1}$

x. R is said to be right (left) weakly π-regular [10] if for every $a \in R$, there exists a positive integer n such that $a^n \in a^nRa^nR$ ($a^n \in Ra^nRa^n$) and it is called weakly π-regular if it is both right and left weakly π-regular. A ring R is said to be right (left) sπ-weakly regular [1] if for every $a \in R$, there exists a positive integer n such that $a^n \in a^nRa^{2n}R$ ($a^n \in Ra^{2n}Ra^n$) and it is called sπ-weakly regular if it is both right and left sπ-weakly regular. A ring R is said to be π-biregular [22] if for any $a \in R$, Ra^nR is generated by a central idempotent, for some positive integer n. A right R-module M is called right principally injective (briefly, right P-injective) [15] if for any principal right ideal aR of R and any right R-homomorphism of aR into M extends to one of R into M. A ring R is called right P-injective if the right R-module R_R is P-injective. A right R-module M is called Generalized P-injective (briefly, GP-injective) [18] if for any $a \in R$, there exists a positive integer n and any right R-homomorphism of a^nR into M extends to one of R into M. A ring R is called right (left) GP-injective if the right (left) R-module R_R (${}_RR$) is GP-injective. A right R-module M is called YJ-injective [20] if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^nR into M extends to one of R into M. A ring R is called right (left) SGPI-ring [1] (GP-V-ring) [29] if every simple right (left) R-module is YJ-injective. A ring R is called right (left) GP-V′-ring [29] if every simple singular right (left) R-module is YJ-injective.

II. Generalized Flatness and Generalized SF-Rings

The concept of generalized flatness and generalized SF-rings were first introduced in [1]. Now, in this section we continue to study generalized flatness and generalized SF-rings.

We start this section with the following definitions.

Definition 2.1 [1]:

Let R be a ring and I be a right (left) ideal of R. Then R / I is a right (left) generalized flat if and only if for each $a \in I$, there exists $b \in I$ and a positive integer n such that $a^n = ba^n$ ($a^n = a^nb$).

Definition 2.2 [1]:

A ring R is called a right (left) generalized SF-ring if every simple right (left) R-module is a right (left) generalized flat.

Definition 2.3 [26]:

Let M be a left R-module and I a left ideal of R. Then M is I-complete if any left R-homomorphism from I to M can be extended to a left R-homomorphism from R to M.

Also M is PR-complete or P-injective if M is Ra-complete, for any principal left ideal Ra of R.

Theorem 2.4:

Let I be an ideal of a ring R. If R / I is left R-generalized flat and J is any right ideal of R such that $J \subseteq I$ or $J + I = R$, then R / I is J-complete.

Proof:

Let $f : J \to R / I$ be any right R-homomorphism. Let $J \subseteq I$ and $x \in J$. Since ${}_R(R / I)$ is generalized flat, there exists some $y \in I$ and a positive integer n such that $x^n = x^ny$. Then, $f(x^n) = f(x^ny) = f(x^n)y = 0$ as y in I. Hence $f = 0$ and f can be trivially extended to a right R-homomorphism $R \to R / I$. If $J + I = R$, then there exists some $b \in J$ and $a \in I$ such that $b + a = 1$. Therefore, for all $x \in J$, $bx + ax = x$. So, $f(x) = (bx) + f(ax)$. Since $ax \in I$ and ${}_R(R / I)$ is generalized flat, there exists some $t \in I$ and a positive integer n such that $(ax)^n = (ax)^n t$. Since $(ax)^n$ is in J, we get $f(a^nx^n) = f(a^nx^nt) = f(a^nx^n)\,t = 0$. Therefore, $f(x^n) = f(bx^n) = f(b)x^n$. Hence f can be extended to a right R-homomorphism $R \to R / I$. ♦

Definition 2.5:

Let M be a left R-module and Ra^n a principal left ideal of R, for any $a \in R$ and a positive integer n. Then M is Ra^n-complete if any left R-homomorphism from Ra^n to M can be extended to a left R-homomorphism from R to M.

Also, M is GPR-complete or GP-injective if M is Ra^n-complete, for any principal left ideal Ra^n of R.
The following result is a relation between GPR-complete with left R-generalized flat of a quoient ring.

Theorem 2.6:

Let L be a maximal right ideal of R, which is two-sided. If R / L is GPR-complete, then R / L is left R-generalized flat.

Proof:

It is sufficient to prove that $a^n \in a^nL$, for all $a \in L$ and a positive integer n. Consider, the epimorphism $f: R / L \to a^nR / a^nL$ defined by $f(t+L) = a^nt + a^nL$, for all $t \in R$ and a positive integer n. If $f = 0$, then $a^nR = a^nL$. Then $a^n \in a^nL$. Suppose that $f \neq 0$. Then Ker $f = \{0\}$ (since R / L is a simple right R-module) and so f is an isomorphism. Therefore, $_R(R / L)$ is a^nR-complete implies that a^nR / a^nL is a^nR-complete. Thus, the right R-homomorphism g: $a^nR \to a^nR / a^nL$ defined by $g(a^nt) = a^nt + a^nL$, for all $t \in R$ can be extended to a right R-homomorphism h: $R \to a^nR / a^nL$. So, $a^n + a^nL = g(a^n) = h(a^n) = h(1)a^n = (a^nt + a^nL)a^n$, for some $t \in R$. That is, $a^n - a^nta^n \in a^nL$. But $a^nta^n \in a^nL$, so that $a^n \in a^nL$. This completes the proof. ♦

Theorem 2.7:

Let L be a maximal right ideal of R, which is two-sided, then the following are equivalent:

(1) R / L is left R-generalized flat.

(2) R / L is a GP-injective right R-module.

(3) R / L is GPR-complete.

Proof:

(1) ⇒ (2).

Let R / L be a left R-generalized flat. Let $(0) \neq I$ be a right ideal of R. Then, L + I = R. Thus, R / L is L-complete by Theorem 2.4. Therefore, R / L is a GP-injective right R-module.

(2) ⇒ (3). It is obvious.

(3) ⇒ (1). Follows from Theorem 2.6. ♦

Theorem 2.8:

Let R be a left quasi-duo, right generalized SF-ring. Then, R is a left GP-V- ring.

Proof:

Let S be a simple left R-module. Then, $S \cong R / L$, for some maximal left ideal L of R. Since R is left quasi-duo, then L is a two-sided ideal of R. As R is right generalized SF-ring, $(R / L)_R$ is generalized flat. So, by Theorem 2.7, $_R(R / L)$ is GP-injective. Hence, S is GP-injective. So, R is a left GP-V-ring. ♦

Theorem 2.9:

Let R be a left quasi-duo, right generalized SF-ring. Then, R is a left GP-injective ring.

Proof:

Let S be a simple left R-module. Then, $S \cong R / L$, for some maximal left ideal L of R. Since R is a left quasi-duo, then L is a two-sided ideal of R. Since R is right generalized SF-ring, then $(R / L)_R$ is generalized flat. So, by Theorem 2.8, R is a left GP-V-ring. It means that $_R(R / L)$ is GP-injective. Hence S is GP-injective. So, R is a left GP-injective. ♦

Theorem 2.10:

If I is a GP-injective left ideal of a ring R, then $_R(R / I)$ is generalized flat and hence R is a left generalized SF-ring.

Proof:

Let $a \in I$. Consider the inclusion map j: $Ra^n \to I$, for some positive integer n. Since I is GP-injective, j can be extended to $f: R \to I$. So, $a^n = j(a^n) = f(a^n) = a^n f(1) = a^nb$, where $b = f(1) \in I$. Therefore, $_R(R / I)$ is generalized flat and hence R is a left generalized SF-ring. ♦

Lemma 2.11[2]:

If R is reduced, then $\ell(a^n) = r(a^n)$, for any $a \in R$ and a positive integer n.

Theorem 2.12 ([1, Theorem 5.1.14]):

Let R be a ring such that for every $a \in R$ and a positive integer n, $\ell(a^n)$ is a right ideal. If R is a right generalized SF-ring, then R is π-regular.

Corollary 2.13:

A reduced right generalized SF-ring R is strongly π-regular.

Proof:

Since R is reduced, then by Lemma 2.11, $\ell(a^n) = r(a^n)$, for every $a \in R$ and a positive integer n. Therefore, by Theorem 2.12, R is π-regular. This implies R is strongly π-regular, as R is reduced. ♦

Following [21], a ring R is called ERT-ring if every essential right ideal of R is a two-sided ideal.

Lemma 2.14[24]:

If R is an ERT-ring, then R / Y(R) is reduced.

Proposition 2.15[16]:

For a ring R. If $Y(R) \neq (0)$ ($Z(R) \neq (0)$), then there exists $0 \neq y \in Y(R)$ ($0 \neq y \in Z(R)$) such that $y^2 = 0$.

Theorem 2.16:

Let R be a ring. The following are equivalent:

(1) R is strongly π-regular.

(2) R is a right generalized SF-ring and ERT-ring.

Proof:

(1) $\Rightarrow$ (2). It is obvious.

(2) $\Rightarrow$ (1). By Lemma 2.14, R / Y(R) is a reduced ring. We claim that Y(R) = (0). Suppose that $Y(R) \neq (0)$, then by Proposition 2.15, there exists $0 \neq y \in Y(R)$ such that $y^2 = 0$. Let N be a maximal right ideal containing $r(y^n)$. Since $r(y^n)$ is an essential two-sided ideal of R, then N must be an essential two-sided ideal of R. On the other hand, since R / N is generalized flat and $y \in N$, there exists $c \in N$ and a positive integer n such that $y^n = y^n c$, whence $1-c \in r(y^n) \subseteq N$, yielding $1 \in N$, which is contradicts $N \neq R$. This proves that R is reduced ring.

In order to show that R is π-regular, we need to prove that $a^nR + r(a^n) = R$, for any $a \in R$ and a positive integer n. Suppose that $a^nR + r(a^n) \neq R$, then there exists a maximal right ideal L containing $a^nR + r(a^n)$. But $a \in L$ and R / L is generalized flat, there exists $b \in L$ and a positive integer n such that $a^n = ba^n$, whence $1-b \in \ell(a^n) = r(a^n) \subseteq L$, yielding $1 \in L$, which contradicts $L \neq R$. In particular, $a^nt + d = 1$, for some $t \in R$ and $d \in r(a^n)$. Whence, $a^nta^n = a^n$. This proves that R is strongly π-regular. ♦

Lemma 2.17 [7]:

Let R be a ring, then $a \in J(R)$ if and only if 1 - at is invertible in R, for each $t \in R$.

Recall that a ring R is a right uniform [8] if every right ideal of R is essential.

Theorem 2.18:

Let R be a right generalized SF-ring.

(1) If $\ell(a^n) = (0)$, for some positive integer n, then a is a right invertible.

(2) If J(R) is reduced, then J(R) is a nil ideal of R.

(3) If R is a right uniform ring, then R is a division ring.

(4) $Z(R) \subseteq J(R)$.

Proof:

(1) Let $a \in R$ with $\ell(a^n) = (0)$, for some positive integer n. If $a^nR \neq R$, there exists a maximal right ideal L containing a^nR. Since $a \in L$ and R / L is generalized flat, there exists $b \in L$ and a positive integer n such that $a^n = ba^n$. Whence, $1- b \in \ell(a^n) = (0)$, yielding $1 \in L$, which contradicts $L \neq R$. Therefore, $a^nR = R$. Then, $a^nt = 1$ and hence a^n is a right invertible in R [1, Theorem 5.1.12]. Therefore, $a(a^{n-1} t) = 1$. If we set $s = a^{n-1} t \in R$, then as = 1, a is a right invertible.

(2) Let $a \in J(R)$, then by Corollary 2.13, J(R) is strongly π-regular and hence there exists $b \in J(R)$ such that $a^n = a^{n+1}b$. But $a \in J(R)$, then by Lemma 2.17, (1-ab)u = 1, for some $u \in R$, this implies that $(a^n - a^{n+1}b)\,u = a^n$. Thus, $a^n = 0$ and hence J(R) is a nil ideal of R.
(3) Suppose that $Y(R) \neq (0)$, then there exists a maximal right ideal L containing Y(R). For any $0 \neq y \in Y(R)$, gives $y \in L$, but R / L is generalized flat, then there exists $x \in L$ and a positive integer n such that $y^n = xy^n$, whence $y^n \in r(1\text{-}x)$. On the other hand, since R is a right uniform, then r(1-x) is an essential right ideal of R. Thus, $1\text{-}x \in Y(R) \subseteq L$, this implies that $1 \in L$, contradicting $L \neq R$. Therefore, Y(R) = (0). On the other hand, since R is uniform ring, then for every $a \in R$, $r(a^n) = (0)$, then by (1) R is a division ring.
(4) Let $x \in Z(R)$, then for any $t \in R$, we have $\ell(1\text{-}xt) = (0)$, which implies that (1-xt) is right invertible, so that $x \in J(R)$. Whence, $Z(R) \subseteq J(R)$. ♦

Definition 2.19 [26]:

Let I be a right ideal of a ring R. Then, the set $\bar{I} = \{x \in R: xI \subseteq I\}$ is the idealizer of I in R.

Theorem 2.20:

Let L be a maximal right ideal of R. Then, R / $\bar{I}$ is right R-generalized flat if and only if $\bar{I}$ / I is left $\bar{I}$GP-injective if and only if $\bar{I}$ / I is $\bar{I}$GP-complete.

Proof:

It is easy to see that R / I is right R-generalized flat if and only if $\bar{I}$ / I is right $\bar{I}$-generalized flat. Also, we note that I is a maximal left ideal of $\bar{I}$, which is two-sided. Hence applying the left analogue of Theorem 2.7, the proof can be completed. ♦

Corollary 2.21:

If M a simple left R-module over a commutative ring R, then M is generalized flat if and only if M is GP-injective if and only if M is GPR-complete.

Proof:

Directly it is true. ♦

Theorem 2.22:

Let R be a left quasi-duo left generalized SF-ring. Then R is a right GP-V-ring.

Proof:

Let A = R / J(R). Since R is a left generalized SF-ring, A is a left generalized SF-ring. Also, R is left quasi-duo implies A is left quasi-duo. A gain A is semi-primitive, so A is reduced. Therefore, A is strongly π-regular and hence is right quasi-duo. Therefore, by duality of Theorem 2.8, R is a left GP-V-ring. ♦

Theorem 2.23:

Let R be a generalized SF-ring. If $\ell(a^n) \subseteq r(a)$, for each $a \in J(R)$ and a positive integer n, then J(R) = (0).

Proof:

Let $0 \neq a \in J(R)$. If $RaR + r(a) \neq R$, then there exists a maximal right ideal L of R such that $RaR + r(a) \subseteq L$. Since R is a generalized SF-ring, then R / L is right generalized flat. Therefore, there exists $b \in R$ and a positive integer n such that $a^n = ba^n$. This implies that $(1\text{-}b) \in \ell(a^n) \subseteq r(a) \subseteq L$, then we obtain $1 \in L$, this is a contradiction. Therefore, RaR + r(a) = R. Then, 1 = x + c, for some $x \in RaR$ and $c \in r(a)$. This implies that a = ax + ac = ax. Therefore, a(1-x) = 0. Since $x \in RaR \subseteq J(R)$, then by Lemma 2.17, (1 - x) is invertible. Therefore, a = 0. This is a contradiction with $a \neq 0$. Whence, J(R) = (0). ♦

III. Main Results

In this section several results on generalized flatness and generalized SF-rings will be found. Moreover, we compare them with regular rings, left non-singular, strongly regular, strongly π-regular, π-biregular and sπ-weakly regular rings under some conditions.

We start this section with the following result.

Theorem 3.1:

Let each principal right ideal a^nR of a ring R be projective, for each $a \in R$ and a positive integer n and I is an ideal of R such that R / I is GPR-complete. Then, R / I is left R-generalized flat. Further, if R is reduced, then R / I will be right R-generalized flat.

Proof:

Let $a \in I$ and $g: a^nR \to a^nR / a^nI$ be a natural map, for some positive integer n. Let $f: R / I \to a^nR / a^nI$ be a homomorphism defined by $f(t + I) = a^nt + a^nI$, for some positive integer n. Clearly f is an epimorphism. Since a^nR is projective, there exists a homomorphism $h: a^nR \to R / I$ such that $fh = g$. Since R / I is a^nR-complete, h extends to a homomorphism $\bar{h}: R \to R / I$. Let $\bar{g} = f\bar{h}$. Now, for all $x \in a^nR$, $\bar{g}(x) = f(\bar{h}(x)) = f(h(x)) = g(x)$. So, $\bar{g}$ extends g. Therefore, $a^n + a^nI = g(a^n) = \bar{g}(a^n) = \bar{g}(1)a^n = (a^nt + a^nI)a^n$, for some $t \in R$ and a positive integer n. So, $(a^n - a^nt a^n) \in a^nI$. Therefore, $a^nta^n \in a^nI$ implies $a^n \in a^nI$. Hence R / I is left R-generalized flat.

Assume that R contains no non-zero nilpotent elements and $a \in I$. Since R / I is left R-generalized flat, there exists some $x \in I$ and a positive integer n such that $a^n = a^nx$. So, $(a^n - xa^n)^2 = a^{2n} - a^nx a^n - xa^{2n} + xa^nxa^n = 0$. Therefore, by assumption, $a^n - xa^n = 0$, which implies $a^n = xa^n$. This implies that R / I is right R-generalized flat. ♦

Corollary 3.2:

If R is a commutative ring in which each principal ideal a^nR is projective, for each $a \in R$ and a positive integer n, then any cyclic module, which is GPR-complete is generalized flat.

Theorem 3.3:

Let R be a ring with every simple right R-module is right GP-injective. Then

(1) For each $a \in R$, there is an $x \in Ra^nR$ such that $a^n = a^nx$, for some positive integer n.

(2) For each ideal I in R, R / I is left R-generalized flat.

(3) For each maximal right ideal L of R, which is two-sided, R / L is right GP-injective.

Proof:

(1) Let $a \in R$. Suppose that $a^n \neq a^nx$, for every $x \in Ra^nR$ and for every positive integer n. Then, $a^n \neq a^nRa^nR$. Let $\Im = \{K: K$ is a right ideal of R, $a^n \notin K$, $a^nRa^nR \subseteq K\}$. Order $\Im$ by inclusion. Then, $\Im \neq \phi$ as $a^nRa^nR \in \Im$. Let C be a chain in $\Im$. Let $\beta = \coprod_{D \in C} D$. Then $\beta \in \Im$ and β is an upper bound of C. Therefore, by Zorn's lemma, $\Im$ has a maximal element K_o. Let $X = (a^nR + K_o) / K_o$. Now, K_o is a maximal submodule of $a^nR + K_o$ [for K_o is a submodule of $a^nR + K_o$ such that $K_o \subseteq K_1 \subset a^nR + K_o$, then $a^n \notin K_1$. Also, $a^nRa^nR \subseteq K_o \subseteq K_1$. So, $K_1 \in \Im$, then K_o is a maximal element of $\Im$ implies $K_1 = K_o$]. Therefore, X is a simple right R-module and hence a^nR is complete by hypothesis. Therefore, the natural map $f: a^nR \to (a^nR + K_o) / K_o$ can be extended to $g: R \to (a^nR + K_o) / K_o$. Then, $a^n + K_0 = f(a^n) = g(a^n) = g(1)a^n = (a^nt + K_0)a^n$, for some $t \in R$, which gives $(a^nR - a^nta^n) \in K_0$. But $a^nta^n \in K_0$ implies $a^n \in K_0$, a contradiction. Therefore, $a^n = a^nx$, for some $x \in a^nRa^n$ and a positive integer n.

(2) Let $a \in I$, from (1), there exists $x \in a^nRa^n \subseteq I$ such that $a^n = a^nx$, for some positive integer n. This implies that R / I is a generalized flat left R-module.

(3) From (2), R / I is a generalized left R-module. This implies that R / L is a GP-injective right R-module by Theorem 2.7. ♦

Theorem 3.4:

The following conditions are equivalent for a reduced ring R.

(1) R is left weakly π-regular.

(2) R is left GP-injective.

(3) R is left SGPI-ring.

(4) R is strongly π-regular.

Proof:

(1) ⇒ (2).

Let S be a simple left R-module. Then, S ≅ R / L, for some maximal left ideal L of R. As R is a left weakly π-regular, then $Rx^n = Rx^nRx^n$, for some positive integer n. So, there exists some $y \in Ry^nR$ such that $x^n = yx^n$. So, R / L is a generalized flat right R-module. Therefore, R / L is a GP-injective left R-module, that is, S is left GP-injective and hence R is a left SGPI-ring.
(2) ⇒ (3). It is obvious.
(3) ⇒ (4).

We shall first show that $a^n = 0$, for every $a \in R$ and a positive integer n. Suppose $Ra^n = (Ra^n)^2$, for some $a \in J(R)$ and a positive integer n, then $0 \neq Ra^n / (Ra^{2n})$ is a finitely generated left R-module and hence has a maximal element $\mu / (Ra^{2n})$. Now, Ra^n / μ is a simple left R-module, so by hypothesis, Ra^n / μ is GP-injective. So, there exists a left R-homomorphism f: $R \rightarrow Ra^n / \mu$, which extends the canonical homomorphism $\dot{\eta}$: $R \rightarrow Ra^n / \mu$. Then $a^n + \mu = \dot{\eta}(a^n) = f(a^n) = a^n f(1) = a^n (ya^n + \mu)$, for some $y \in R$ yielding $a^n + \mu = a^nya^n + \mu$. Hence $(a^ny - 1)a^n \in \mu$. But $a \in J(R)$ implies $a^ny - 1$ is a unit. Therefore, $a^n \in \mu$. This contradicts $Ra^n \neq \mu$. Thus, $Ra^n = (Ra^n)^2$, for all $a \in J(R)$ and a positive integer n. Then, $a^n = \sum_{i=1}^{n} x_i a^n y_i a^n$, for some $x_i \in R$, $y_i \in R$. This implies $(1-\sum_{i=1}^{n} x_i a^n y_i)\, a^n = 0$. Since $a \in J(R)$, $1-\sum_{i=1}^{n} x_i a^n y_i$ is a unit. So, $a^n = 0$. Since R is reduced, then a = 0. Also, by Theorem 3.3, R is a generalized SF-ring. Thus, by Corollary 2.13, R is strongly π-regular.
(4) ⇒ (1). It is obvious. ♦

Recall that a ring R is right (left) weakly continuous [23] if J(R) = Y(R) (J(R) = Z(R)), R / J(R) is regular and idempotent can be lifted modulo J(R).

Lemma 3.5 [23]:

Every regular ring is right (left) weakly continuous.

Theorem 3.6:

If R is a right generalized SF-ring and R has a finite number of maximal right ideals whose product is contained in J(R), then J(R) = (0).

Proof:

Let $N_1, N_2, \ldots, N_m$ be maximal right ideals of R such that $N_1N_2 \ldots N_m \subseteq J(R)$.

First, suppose that J(R) is non-zero reduced. If $x \in J(R)$ and since $x \in N_m$ and R / N_m is generalized flat, then there exist positive integers n_m, $y_m \in N_m$ such that $x^{n_m} = y_m x^{n_m}$, which implies that $1- y_m \in r(x^{n_m})$. Since J(R) is reduced and $x \in J(R)$, then $r(x) = r(x^{n_m})$, thus $1- y_m \in r(x^{n_m}) = r(x)$. Therefore, $x = y_mx$. Since $y_mx \in J(R) \subseteq N_{m-1}$ and R / N_{m-1} is generalized flat, then there exist positive integers n_{m-1}, $y_{m-1} \in N_{m-1}$ such that $x^{n_{m-1}} = y_{m-1} x^{n_{m-1}}$ and we continue this process we get $x = y_{m-1} x$.

Finally, we have $y_i \in N_i$, $1 \leq i \leq m$ such that $y_1 y_2 \ldots y_{m-1} y_m \in N_1 N_2 \ldots N_m \subseteq J(R)$ and $x = y_1 y_2 \ldots y_{m-1} y_m x$. Now, $t(y_1 y_2 \ldots y_{m-1} y_m) = 1$, for some $t \in R$, which yields $x = 1x = t(y_1 y_2 \ldots y_m) x = 0$, which is a contradiction.

Now, suppose that J(R) is not reduced. Then there exists $0 \neq a \in J(R)$ such that $a^2 = 0$. Since $a \in J(R) \subseteq N_m$ and R / N_m is generalized flat, then $a = b_m a$, for some $b_m \in N_m$. Since $b_m a \in J(R) \subseteq N_{m-1}$ and R / N_{m-1} is generalized flat, then $a = b_ma = b_{m-1} b_m a$, for some $b_{m-1} \in N_{m-1}$ and so one. Finally, we have $b_i \in N_i$, $1 \leq i \leq m$ such that $b_1 b_2 \ldots b_m \in N_1 N_2 \ldots N_m \subseteq J(R)$ and $a = b_1 b_2 \ldots b_{m-1} b_m a$.

Now, $u(1 - b_1 b_2 \ldots b_m) = 1$, for some $u \in R$, which yields $a = 1a = u(1 - b_1 b_2 \ldots b_m) a = 0$. Thus, J(R) is reduced and we get J(R) = (0) and by Theorem 2.18 (4), Z(R) ⊆ J(R). Whence, Z(R) = (0). ♦

Corollary 3.7:

Let R be a left weakly continuous, right generalized SF-ring and R has a finite number of maximal right ideals whose product is contained in J(R). Then R is regular.

Proposition 3.8 [16]:

For any R. If Z(R) ∩ Y(R) ≠ (0), then there exists $0 \neq x \in Z(R) \cap Y(R)$ such that $x^2 = 0$.

Remark 3.9 [17]:

If L is an essential right ideal, then R_R / L cannot be projective. We consider the condition (*): R satisfies $\ell(a) \subseteq r(a)$, for any $a \in R$.

Theorem 3.10:

Let R be a ring satisfy condition (*). If every simple right R-module is either generalized flat or projective, then $Z(R) \cap Y(R) = (0)$.

Proof:

First suppose that $Z(R) \cap Y(R)$ is non-zero reduced ideal of R. If $0 \neq x \in Z(R) \cap Y(R)$, r(x) is an essential right ideal of R and $xR \cap r(x) = (0)$. Let $a \in R$ such that $0 \neq xa \in r(x)$. Since $Z(R) \cap Y(R)$ is reduced, then $(xax)^2 = xa\,(x^2a)x = 0$ implies that $xax = 0$, and therefore $(xa)^2 = (xax)a = 0$, which yields $xa = 0$, a contradiction.

Now, suppose that $Z(R) \cap Y(R) = (0)$, then by Proposition 3.8, there exists $0 \neq t \in Z(R) \cap Y(R)$ such that $t^2 = 0$. We will prove that RtR + r(t) = R. If not, let L be a maximal right ideal containing RtR + r(t). Since r(t) is an essential right ideal, then R / L cannot be projective by Remark 3.9, whence it is generalized flat. Since R / L is generalized flat, then there exists $d \in M$ and a positive integer n such that $t^n = dt^n$. Since $t^2 = 0$, then n = 1, so that t = dt and we obtain $1 - d \in \ell(t) \subseteq r(t) \subseteq L$ and $1 \in L$. Whence, L = R contradicts the maximality of L. Therefore, R = RtR + r(t).

Now, 1 = u + v, $u \in RtR$, $v \in r(t)$, which implies that t = tu. Since $u \in Z(R)$ and $Rt \cap \ell(t) = (0)$, then t = 0, a contradiction. Whence, $Z(R) \cap Y(R) = (0)$. ♦

Corollary 3.11:

Let R be a right weakly continuous ring satisfy condition (*). If every simple right R-module is generalized flat or projective, then $Z(R) \cap J(R) = (0)$.

Corollary 3.12:

Let R be a right weakly continuous ring satisfying condition (*). If every simple right R-module is generalized flat or projective, then R is regular.

Proof:

Since R is weakly continuous, then Y(R) = Z(R) = J(R). By Corollary 3.11, $J(R) \cap Y(R) = (0) = J(R) \cap Z(R)$. Hence, J(R) = (0).Whence, R is regular. ♦

Theorem 3.13:

Let R be a semi-prime ring satisfying condition (*), for which every simple right R-module is either generalized flat or projective. Then R is left non-singular.

Proof:

Suppose that $Z(R) \neq (0)$. Then, by Proposition 2.15, there exists $0 \neq y \in Z(R)$ such that $y^2 = 0$. Set L = RyR + r(y). Let K be a complement right ideal of R such that $E = L \oplus K$ is an essential right ideal of R. Then, $KRyR \subseteq K \cap RyR \subseteq K \cap L = (0)$ implies $(RyRK)^2 = (0)$. Since R is semi-prime, then RyRK = (0), which yields $K \subseteq r(y) \subseteq L$. Whence, $K = K \cap L = (0)$. This shows that E = L is an essential right ideal of R. Now, suppose that $L \neq R$. Let N be a maximal right ideal of R containing L. Then, R / N is generalized flat, then there exists $u \in N$ and a positive integer n such that $y^n = uy^n$, which yields n = 1 and $1 - u \in \ell(y) \subseteq r(y) \subseteq N$. Thus, $1 \in N$, contradicting $N \neq R$. Therefore, L = R and 1 = s + t, where $s \in RyR$, $t \in r(y)$ and we have y = ys + yt = ys. Now, $Ry \cap \ell(s) = (0)$ implies that y = 0. This is a contradiction. Whence, R is left non-singular.v ♦

Corollary 3.14:

If R is a semi-prime left weakly continuous ring satisfying condition (*) such that every simple right R-module is either generalized flat or projective, then R is regular.

Recall that a ring R is called FGP-injective [3] if for any $0 \neq a \in R$, there exists $0 \neq c \in R$ such that $0 \neq ac = ca$ and any right R-homomorphism from $ac \in R$ to R extends to an endomorphism of R.

Recall that a ring R is called right (left) Kasch ring [19] if every maximal right (left) ideal of R is a right (left) annihilator.

Lemma 3.15 [3]:

If R is a right Kasch right FGP-injective ring, then J(R) = Y(R) = Z(R).

Lemma 3.16 [25]:

If Y(R) = (0) and $\ell(a) \subseteq r(a)$, for every $a \in R$, then R is reduced.

Theorem 3.17:

Let R be a right Kasch and FGP-injective ring satisfying condition (*) for which every simple right R-module is generalized flat or projective. Then R is strongly regular.

Proof:

Since R is a right Kasch and a right FGP-injective ring, then by Lemma 3.15, Z(R) = J(R) = Y(R). Also, by Theorem 3.10, $Z(R) \cap Y(R) = (0)$, which implies Z(R) = Y(R) = (0). Therefore, by Lemma 3.16, R is reduced. Let $0 \neq a \in R$, we shall prove that aR + r(a) = R. If not, then there exists a maximal right ideal L containing aR + r(a). Since R is a right Kasch ring, then there exists $t \in R$ such that L = r(t). Let x = at + y, where $t \in R$, $y \in r(a)$. So, $x \in aR + r(a) \subseteq r(t)$ and tx = t(at + y) = 0, since ty = 0, then tat = 0. But R is reduced, so we have at = ta = 0, which implies $t \in r(a) \subseteq r(t)$, therefore $t^2 = 0$, since R is reduced, then t = 0, which is a contradiction. So, aR + r(a) = R. Inparticular, 1 = ax + v, where $v \in r(a)$ and $x \in R$, so $a = a^2x$. Whence, R is strongly regular. ♦

To prove the next result we need some statements and definitions as follows:

Proposition 3.18 [8]:

Let R be a ring, then the following statements are equivalent.

(1) R is a strongly regular ring.

(2) R is regular and reduced.

(3) Every principal right ideal of R is generated by a central idempotent.

Recall that a non-zero ideal I of R is called a minimal ideal [7] if there is no ideal J of R such that $0 \subset J \subset I$.

Definition 3.19 [28]:

Let R be a ring. An element $k \in R$ is called a minimal element if kR is a minimal ideal of R.

An element $e \in R$ is called right minimal idempotent if e is a minimal element and $e^2 = e$.

An idempotent $e \in R$ is called semicentral if ea = eae, for each $a \in R$.

Definition 3.20 [28]:

A ring R is called right min-abel if every minimal right idempotent element of R is right semicentral.

Proposition 3.21 [28]:

Let R be a right min-abel, then R is MERT-ring if and only if R is right quasi-duo.

Proposition 3.22 [30]:

If R is right (left) quasi-duo and J(R) = (0), then R is reduced.

Theorem 3.23:

Let R be a ring satisfies $\ell(a) \subseteq r(a)$, for each $a \in J(R)$. Then, the following statements are equivalent:

(1) R is a strongly regular ring.

(2) R is right min-abel, MERT and a right generalized SF-ring.

Proof:

(1) $\Rightarrow$ (2). Follows from Proposition 3.18 and Proposition 3.21.

(2) $\Rightarrow$ (1). Suppose that there exists $0 \neq x \in J(R)$ such that $x^2 = 0$. Let I = RxR + r(x). If $I \neq R$, then there exists a maximal right ideal Q of R such that $I \subseteq Q$. Since R is a right generalized SF-ring, then R / Q is right generalized flat, then there exists $d \in R$ and a positive integer n such that $x^n = dx^n$. Since $x^2 = 0$, then we get n = 1 and hence x = dx. This implies that $(1\text{-}d) \in \ell(x) \subseteq r(x) \subseteq Q$. Therefore, $1 \in Q$, this is a contradiction since $I \neq R$. In particular, 1 = s + t, for some $s \in RxR$ and $t \in r(x)$. This implies that x = xs. Since $s \in J(R)$, then by Lemma 2.17, 1-s is invertible in R. Therefore, x = 0, this is a contradiction. Whence,

J(R) is a reduced ideal of R. Now, suppose that $0 \neq y \in J(R)$. Since J(R) is a reduced ideal, then for any positive integer m, we have $\ell(y^m) = \ell(y) = r(y) = t(y^m)$.

We set P = RyR + r(y). If $P \neq R$, then there exists a maximal right ideal N of R such that $P \subseteq N$. We use the same method in Theorem 2.23, we get J(R) = (0). Then by Proposition 3.21, R is right quasi-duo. Therefore, by Proposition 3.22, R is reduced.

Suppose that there exists an element $a \in R$ such that $aR + r(a) \neq R$. Then, there exists a maximal right ideal L of R such that $aR + r(a) \subseteq L$. Since R is a right generalized SF-ring, then R / L is right generalized flat. Then there exists $c \in L$ and a positive integer n such that $a^n = ca^n$. This implies that $(1 - c) \in \ell(a^n) = r(a^n) \subseteq L$. Therefore, $1 \in L$. This is a contradiction since $L \neq R$. Therefore, aR + r(a) = R. In particular, ab + t = 1, for some $b \in R$ and $t \in r(a)$. This implies that $a^2b = a$. Whence, R is a strongly regular ring. ♦

Definition 3.24 [26]:

An additive subgroup L of a ring R is a weakly left ideal of R if for every $x \in L$ and for every $t \in R$, there exists a natural number n such that $(tx)^n \in L$. The notation of a weakly right ideal of a ring R is defined similarly.

Recall that a ring R is an LW-ring (RW-ring) [26] if every left (right) ideal of R is a weakly right (left) ideal of R.

Theorem 3.25 ([12], Lemma 2):

Let R be a semiprimitive ring. If every left annihilator of an elemenet of R is a weakly right ideal of R, then R is reduced.

Theorem 3.26:

The following statements are equivalent for a ring R:

(1) R is a strongly π-regular ring.

(2) R is an LW left generalized SF-ring.

(3) R is an LW right generalized SF-ring.

Proof:

(1) ⇒ (2) and (1) ⇒ (3) are obvious.

(2) ⇒ (1). R / J(R) being a semi-primitive LW-ring is reduced by Theorem 3.25. Also, R is a left generalized SF-ring implies R / J(R) is a left generalized SF-ring. Therefore, by Corollary 2.13, R / J(R) is strongly π-regular. Let L be a maximal left ideal of R. Then, L / J(R) is a left ideal of R / J(R) and thus L / J(R) is an ideal of R / J(R) (since a strongly π-regular ring is left and right duo). Thus, L is an ideal of R. Therefore, R is left quasi-duo. Thus, R is strongly π-regular. Hence (2) implies (1).

(3) ⇒ (1) can be proved similarly. ♦

Theorem 3.27 [9]:

If R is a ZI, left GP-V′-ring, then R is reduced.

Recall that a ring R is called a quasi ZI-ring [29] if for any non-zero elements $a, b \in R$, ab = 0 implies that there exists a positive integer n such that $a^n \neq 0$ and $a^nRb^n = 0$.

Proposition 3.28 ([29], Proposition 3.1):

If R is a quasi ZI, left GP-V′-ring, then R is reduced.

Theorem 3.29:

Let R be a quasi ZI, left GP-V′-ring. If R is a right generalized SF-ring. Then, R is a π-biregular ring.

Proof:

Since R is a quasi ZI and left GP-V′-ring. Then by Proposition 3.28, R is reduced.

Now, for any $0 \neq t \in R$ and a positive integer n, $r(Rt^nR) = \ell(Rt^nR) = \ell(Rt^n) = r(t^nR) = r(t^n)$. If $E = Rt^nR + r(t^n)$ and also $Rt^nR \cap r(Rt^nR) = (0)$, then $E = Rt^nR \oplus r(Rt^nR)$.

Let $E \neq R$, then there exists a maximal right ideal L of R such that $E \subseteq L$. Since R is a right generalized SF-ring, then R / L is right generalized flat and $t^n \in L$, there exists $b \in L$ and a positive integer m such that $(t^n)^m = b(t^n)^m$. Now, $(1-b) \in \ell((t^n)^m) = r(t^n) \subseteq L$. Then, $1 \in L$, this is a contradiction. Now, $R = E = Rt^nR \oplus$

$r(Rt^nR)$. Since every idempotent element in a reduced ring is central. Then, Rt^nR is generated by a central idempotent. Whence, R is a π-biregular ring. ♦

Corollary 3.30:

Let R be a ZI, left GP-V′-ring. If R is a right generalized SF-ring, then R is π-biregular.

Proof:

Follows from Theorem 3.27 and by using the same method as in Theorem 3.29, we get the result. ♦

Finally, we give the following result.

Theorem 3.31:

Let R be a quasi ZI and left GP-V′-ring. If every simple right R-module is either YJ-injective or a generalized SF-ring, then R is a right sπ-weakly regular ring.

Proof:

To prove that R is a right sπ-weakly regular ring. We prove that $Rt^{2n}R + r(t^{2n}) = R$, for each $t \in R$ and a positive integer n. suppose that there exists $b \in R$ such that $Rb^{2n}R + r(b^n) \neq R$, for all positive integer n. Then, there exists a maximal right ideal L of R such that $Rb^{2n}R + r(b^n) \subseteq L$. Therefore, R / L is either YJ-injective or generalized flat.

First suppose that R / L is YJ-injective, then there exists a positive integer m such that $(b^n)^m \neq 0$ and any right R-homomorphism from $(b^n)^mR$ to R / L can be extended to R into R / L.

Now, we define a right R-homomorphism $f : (b^n)^mR \rightarrow R / L$ as $f(b^{nm}x) = x + L$, for all $x \in R$.

Now, f is a well-defined because R is a quasi ZI ring. Therefore, $1 + L = f(b^{nm}) = yb^{nm} + L$, $y \in R$. Then, $(1-yb^{nm}) \in L$. Since $y(b^{nm}) \in Rb^{2n}R \subseteq L$, then $1 \in L$, this is a contradiction since $L \neq R$. Next, R / L is a right generalized SF-ring. Therefore, $b^n \in L$, then there exists $x \in L$ and a positive integer m such that $(b^n)^m = x(b^n)^m$. Therefore, $(1-x) \in \ell(b^{nm})$. But R is a quasi ZI and a left GP-V′-ring, then by Proposition 3.28, R is reduced. Therefore, $(1-x) \in \ell(b^{nm}) = r(b^n) \subseteq L$. This implies that $1 \in L$, this is also a contradiction. Therefore, $Rt^{2n}R + r(t^n) = R$. If we set $ct^{2n}d + x = 1$, for some $x \in r(t^n)$, $c, d \in R$. Then, $t^n = t^nct^{2n}d$. Whence, R is a right sπ-weakly regular ring. ♦

Corollary 3.32:

Let R be a ZI and a left GP-V′-ring. If every simple right R-module is either YJ-injective or a generalized SF-ring, then R is a right sπ-weakly regular ring.

Proof:

Follows from Theorem 3.31 and Theorem 3.27. ♦

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