



Restricted Detour Polynomials of Straight Chains of Graphs

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Abstract

Recurrence relations for the restricted detour polynomials of straight chains of vertex identified and edge-introducing graphs are obtained. Then, restricted detour polynomials for chains of hexagons and pentagons are also obtained with their restricted detour indices.

Introduction

In this paper, we consider finite connected graphs without loops or multiple edges. For concepts and notations undefined here, see [2] and [3].

The **restricted detour distance** between vertices u and v in a connected graph G , denoted by $D^*(u, v)$ or $D_G^*(u, v)$ is the length of a longest $u - v$ path P such that $\langle V(P) \rangle = P$, [4]. It is clear that $D^*(u, v) = 0$ if and only if $u = v$, and $D^*(u, v) = 1$ if and only if uv is an edge of G . The **restricted detour diameter** of a connected graph G is denoted by $Diam^*G$ and defined as

$$Diam^*G = \max_{u,v} \{D^*(u, v)\}.$$

An induced $u - v$ path of length $D^*(u, v)$ is called a **restricted (or an induced) detour path** [6]. Moreover, a connected graph G is called a **restricted detour graph** if $D^*(u, v) = d(u, v)$ for every pairs u, v of vertices in G . It is clear that all trees, complete graphs, and complete bipartite graphs are restricted detour graphs. The Wiener index of G with respect to restricted detour distance is denoted by $dd^*(G)$ and defined by

$$dd^*(G) = \sum_{u,v} D_G^*(u, v),$$

and called **restricted detour index** of G [6].

The **restricted detour polynomial** of G , denoted by $D^*(G; v)$ is defined as follows [1, 6]:

$$D^*(G; x) = \sum_{u,v} x^{D^*(u,v)}.$$

One may easily notice that

$$D^*(G; x) = \sum_{k=0}^{\delta^*} C^*(G, k)x^k,$$

in which $C^*(G, k)$ is the number of unordered pairs of vertices u, v of G such that $D_G^*(u, v) = k$, and δ^* is the restricted detour diameter of G .

Let u be a vertex of G , and let $C^*(u, G; k)$ be the number of vertices v of G such that $D^*(u, v) = k$, then the polynomial defined by

$$D^*(u, G; x) = \sum_{k \geq 0} C^*(u, G; k)x^k,$$

is called the **restricted detour polynomial of vertex u** .

It is clear that

$$D^*(G; x) = \frac{1}{2} \left(\sum_{u \in V(G)} D^*(u, G; x) + p \right),$$

where p is the order of G .

One may easily notice that

$$dd^*(G) = \frac{d}{dx} D^*(G; x) |_{x=1}.$$

In [1] and [6] authors obtained restricted detour polynomials and indices of some compound graphs. In this paper, we obtain restricted detour polynomials for some other graphs, namely, straight chains of vertex identified graphs and edge introducing graphs.

Chains of Vertex Identified Graphs

For disjoint graphs G_1 and G_2 , let u be a vertex of G_1 and v be a vertex of G_2 , then the **vertex-identifying graph** $G_1 \bullet G_2$ is the graph constructed from G_1 and G_2 by identifying u and v . And, the **edge-introducing graph** $G_1 : G_2$ is the graph constructed from G_1 and G_2 by adding a new edge joining u and v . In 1993, Gutman [5] obtained $H(G_1 \bullet G_2; x)$ and $H(G_1 : G_2; x)$ in terms of the Hosoya polynomials of G_1 and G_2 with respect to (standard) distance. With respect to restricted detour distance, we have the following theorem:

Theorem 2.1.[1] Let u be a vertex of G_1 and v a vertex of G_2 , then

- (1) $D^*(G_1 \bullet G_2; x) = D^*(G_1; x) + D^*(G_2; x) + D^*(u, G_1; x)D^*(v, G_2; x) - D^*(u, G_1; x) - D^*(v, G_2; x).$
- (2) $D^*(G_1 : G_2; x) = D^*(G_1; x) + D^*(G_2; x) + xD^*(u, G_1; x)D^*(v, G_2; x).$ ■

Let G_1, G_2, \dots, G_m be pairwise disjoint graphs, each is connected, and let $p_i \geq 2$ be the order of G_i for $i = 1, 2, \dots, m$. Moreover, let u_i, v_i be distinct vertices of G_i . Define the **straight chain of identifying graphs** \mathcal{G}_m as the graph constructed from G_1, G_2, \dots, G_m by identifying vertices u_{i+1} and v_i for all $i = 1, 2, \dots, m - 1$, that is

$$\mathcal{G}_m = \mathcal{G}_{m-1} \bullet G_m = (\mathcal{G}_{m-2} \bullet G_{m-1}) \bullet G_m = G_1 \bullet G_2 \bullet \dots \bullet G_{m-1} \bullet G_m. \tag{2.1}$$

The graph \mathcal{G}_m is shown in Fig.2.1, where $v_i = u_{i+1}$.

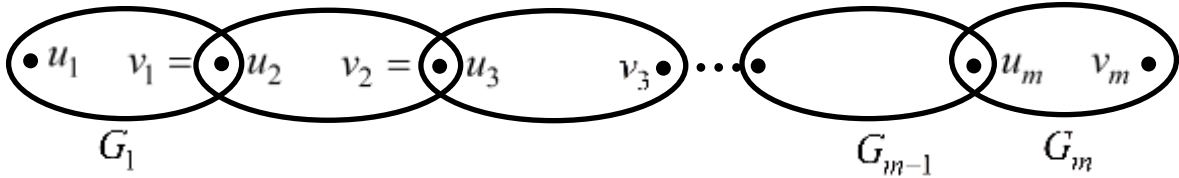


Fig.2.1. The straight chain graph \mathcal{G}_m

It is clear that

$$p(\mathcal{G}_m) = \sum_{i=1}^m p(G_i) - (m - 1),$$

$$q(\mathcal{G}_m) = \sum_{i=1}^m q(G_i).$$

A recurrence relation for obtaining $D^*(\mathcal{G}_m; x)$ is given in the following theorem:

Theorem 2.2. For $m \geq 2$,

$$D^*(\mathcal{G}_m; x) = D^*(\mathcal{G}_{m-1}; x) + D^*(G_m; x) + D^*(u_m, G_m; x)D^*(v_{m-1}, \mathcal{G}_{m-1}; x) - D^*(u_m, G_m; x) - D^*(v_{m-1}, \mathcal{G}_{m-1}; x), \quad (2.2)$$

where

$$D^*(v_{m-1}, \mathcal{G}_{m-1}; x) = \sum_{i=1}^{m-1} \left\{ x^{\sum_{j=i+1}^{m-1} \gamma(j)} (D(v_i, G_i; x) - 1) \right\} + 1, \quad (2.3)$$

in which

$$\gamma(j) = D_{G_j}(u_j, v_j).$$

Proof. Since $\mathcal{G}_m = \mathcal{G}_{m-1} \bullet G_m$, then by *Theorem 2.1(1)*, we get (2.2). Thus, we need to determine $D^*(v_{m-1}, \mathcal{G}_{m-1}; x)$. Let w be any vertex of \mathcal{G}_{m-1} . Suppose $w \in V(G_i)$ for $i = 1, 2, \dots, m - 1$. Then, from Fig.2.1,

$$D_{\mathcal{G}_{m-1}}^*(w, v_{m-1}) = D_{G_i}^*(w, v_i) + \sum_{j=i+1}^{m-1} D_{G_j}^*(u_j, v_j).$$

The polynomial corresponding to this distance from v_{m-1} , is

$$F_i(w; x) = x^{D_{G_i}(w, v_i) + \sum_{j=i+1}^{m-1} D_j(u_j, v_j)} = x^{D_{G_i}(w, v_i)} x^{\sum_{j=i+1}^{m-1} \gamma(j)}.$$

Thus, summing for all w in G_i , we get

$$F_i(x) = x^{\sum_{j=i+1}^{m-1} \gamma(j)} \sum_{w \in V(G_i)} x^{D_{G_i}(w, v_i)} = x^{\sum_{j=i+1}^{m-1} \gamma(j)} [D^*(v_i, G_i; x) - 1],$$

with taking v_i as a vertex of G_{i+1} .

Now, summing up for $i = 1, 2, \dots, m - 1$, and adding 1, corresponding to the distance form v_{m-1} to itself, we get the formula given in (2.3). ■

To illustrate the use of the formula for $D^*(v_{m-1}, \mathcal{G}_{m-1}; x)$, we give the following example:

Example. Consider the graph \mathcal{G}_4 shown in Fig.2.2. We can find

$$\begin{aligned} \gamma(2) &= 2 \text{ and } \gamma(3) = 3, \\ D^*(v_1, G_1; x) &= 1 + 2x + x^3 + 2x^4, \\ D^*(v_2, G_2; x) &= 1 + 2x + x^2, \\ D^*(v_3, G_3; x) &= 1 + 2x + 2x^3. \end{aligned}$$

Thus

$$\begin{aligned}
 D^*(v_3, \mathcal{G}_3; x) &= x^{\sum_{j=2}^3 \gamma_j} [2x + x^3 + 2x^4] + x^{\sum_{j=3}^3 \gamma_j} [2x + x^2] \\
 &\quad + x^{\sum_{j=4}^3} [2x + 2x^3] + 1 \\
 &= x^5(2x + x^3 + 2x^4) + x^3(2x + x^2) + x^0(2x + 2x^3) + 1 \\
 &= 2x^9 + x^8 + 2x^6 + x^5 + 2x^4 + 2x^3 + 2x + 1.
 \end{aligned}$$

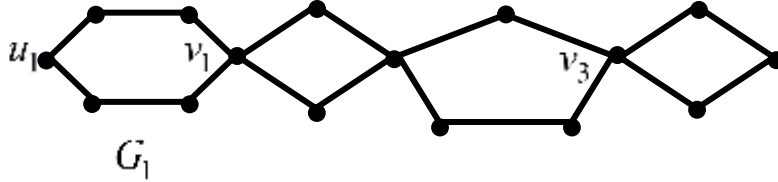


Fig. 2.2.

Let G_1, G_2, \dots, G_m be m copies of a connected graph G of order $p \geq 2$ with a pair of fixed distinct vertices u and v as their identified vertices in the corresponding copies. Such straight chains of identified vertices is denoted by $\mathcal{G}_m(G)$, and $D^*(v, G; x)$ is the same for each copy, and for all j ,

$$\gamma(j) = D_{G_i}(u_i, v_i) = D_G(u, v).$$

Therefore,

$$\sum_{j=i+1}^{m-1} \gamma(j) = \sum_{j=i+1}^{m-1} D_G(u, v) = (m - i - 1)D_G(u, v).$$

Thus, by Theorem 2.2,

$$D^*(v_{m-1}, \mathcal{G}_{m-1}(G); x) = \sum_{i=1}^{m-1} x^{(m-i-1)D_G(u,v)} [D^*(v_i, G_i; x) - 1] + 1.$$

Hence, we have the following result:

Theorem 2.3. For $m \geq 2$,

$$\begin{aligned}
 D^*(\mathcal{G}_m(G); x) &= D^*(\mathcal{G}_{m-1}(G); x) + D^*(G; x) - D^*(u, G; x) + [D^*(u, G; x) \\
 &\quad - 1] \left\{ [D^*(v, G; x) - 1] \sum_{i=1}^{m-1} x^{(m-i-1)D_G(u,v)} + 1 \right\}. \blacksquare
 \end{aligned}$$

In Theorem 2.3, suppose that G is a hexagon C_6 , and let u and v are diametrical vertices in C_6 , then

$$\begin{aligned}
 D_{C_6}^*(u, v) &= 3, \\
 D^*(u, C_6; x) &= D^*(v, C_6; x) = 1 + 2x + x^3 + 2x^4, \\
 D^*(C_6; x) &= 6 + 6x + 3x^3 + 6x^4.
 \end{aligned}$$

Then, by Theorem 2.3,

$$\begin{aligned}
 D^*(\mathcal{G}_m(C_6); x) &= D^*(\mathcal{G}_{m-1}(C_6); x) + (6 + 6x + 3x^3 + 6x^4) - (1 + 2x + x^3 + 2x^4) \\
 &\quad + (2x + x^3 + 2x^4) \left\{ (2x + x^3 + 2x^4) \sum_{i=1}^{m-1} x^{3(m-i-1)+1} \right\}.
 \end{aligned}$$

Solving this recurrence relation, we obtain:

$$\begin{aligned}
 D^*(\mathcal{G}_m(C_6); x) &= D^*(\mathcal{G}_{m-1}(C_6); x) + (5 + 6x + 3x^3 + 6x^4) + (2x + x^3 + 2x^4)^2 \sum_{i=1}^{m-1} x^{3(m-i-1)} \\
 &= D^*(\mathcal{G}_{m-2}(C_6); x) + 2(5 + 6x + 3x^3 + 6x^4) \\
 &\quad + (2x + x^3 + 2x^4)^2 \left\{ \sum_{i=1}^{m-1} x^{3(m-i-2)} + \sum_{i=1}^{m-1} x^{3(m-i-1)} \right\} = \dots \\
 &= D^*(\mathcal{G}_1(C_6); x) + (m-1)(5 + 6x + 3x^3 + 6x^4) + (2x + x^3 + 2x^4)^2 \sum_{j=1}^{m-1} \sum_{i=1}^{m-j} x^{3(m-i-j)}.
 \end{aligned}$$

Hence, we have the following theorem:

Theorem 2.4. For $m \geq 2$,

$$D^*(\mathcal{G}_m(C_6); x) = m(5 + 6x + 3x^3 + 6x^4) + 1 + (2x + x^3 + 2x^4)^2 \sum_{j=1}^{m-1} \sum_{i=1}^{m-j} x^{3(m-i-j)}. \blacksquare$$

Let $G = C_5$, then $\mathcal{G}_m(C_5)$ is a straight chain of m pentagons, (See Fig.2.3.). Then it is clear that

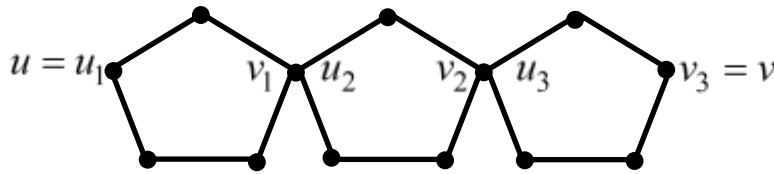


Fig.2.3. The straight chain of 3 pentagons $\mathcal{G}_3(C_5)$.

$$\begin{aligned}
 D^*(u, v) &= 3, \\
 D^*(u, C_5; x) &= D^*(v, C_5; x) = 1 + 2x + 2x^3, \\
 D^*(C_5; x) &= 5 + 5x + 5x^3.
 \end{aligned}$$

Thus, by Theorem 2.3,

$$\begin{aligned}
 D^*(\mathcal{G}_m(C_5); x) &= D^*(\mathcal{G}_{m-1}(C_5); x) + (5 + 5x + 3x^3) - (1 + 2x + 2x^3) \\
 &\quad + (2x + 2x^3)[1 + (2x + 2x^3)] \sum_{i=1}^{m-1} x^{3(m-i-1)} \\
 &= D^*(\mathcal{G}_{m-1}(C_5); x) + (4 + 5x + 5x^3) + (2x + 2x^3)^2 \sum_{i=1}^{m-1} x^{3(m-i-1)}.
 \end{aligned}$$

Solving this recurrence relation, we obtain $D^*(\mathcal{G}_m(C_5); x)$ as given in the next theorem.

Theorem 2.5. For $m \geq 2$,

$$D^*(\mathcal{G}_m(C_5); x) = 1 + m(4 + 5x + 5x^3) + (2x + 2x^3)^2 \sum_{j=1}^{m-1} \sum_{i=1}^{m-j} x^{3(m-i-j)}. \blacksquare$$

Taking the derivatives of $D^*(\mathcal{G}_m(C_6); x)$ and $D^*(\mathcal{G}_m(C_5); x)$ with respect to x and putting $x = 1$, we get the restricted detour indices of $\mathcal{G}_m(C_6)$ and $\mathcal{G}_m(C_5)$ as given in the next result:

Corollary 2.6. For $m \geq 2$,

- (1) $dd^*(\mathcal{G}_m(C_6)) = \frac{5}{2}m^2(5m + 1) - m$,
- (2) $dd^*(\mathcal{G}_m(C_5)) = 8m^3 + 8m^2 + 4m$.

Chains of Edge-Introducing Graphs

Let $G_1, G_2, \dots, G_m, m \geq 2$, be pairwise disjoint graphs, each is connected, and let $p_i \geq 2$. Moreover, let u_i, v_i be distinct vertices of G_i , for $i = 1, 2, \dots, m$. The **straight chain of edge-introducing graphs** \mathcal{J}_m is defined as the graph constructed from G_1, G_2, \dots, G_m by introducing $m - 1$ new edges $u_{i+1}v_i$, for $i = 1, 2, \dots, m - 1$. It is clear that

$$\mathcal{J}_m = \mathcal{J}_{m-1} : G_m = G_1 : G_2 : \dots : G_{m-1} : G_m.$$

The graph $\mathcal{J}_m, m \geq 3$, is shown in Fig.3.1.

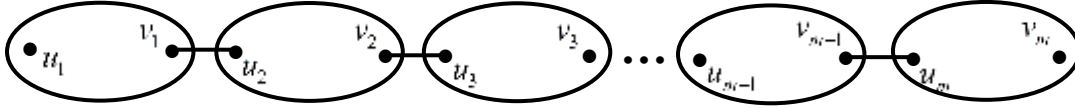


Fig.3.1. The straight chain of edge-introducing graph \mathcal{J}_m .

It is obvious that

$$p(\mathcal{J}_m) = \sum_{i=1}^m p(G_i), \text{ and } q(\mathcal{J}_m) = m - 1 + \sum_{i=1}^m q(G_i).$$

A recurrence relation to obtain $D^*(\mathcal{J}_m; x)$ is given in the following theorem:

Theorem 3.1. For $m \geq 2$,

$$D^*(\mathcal{J}_m; x) = D^*(\mathcal{J}_{m-1}; x) + D^*(G_m; x) + xD^*(u_m, G_m; x)D^*(v_{m-1}, \mathcal{J}_{m-1}; x), \quad (3.1)$$

where

$$D^*(v_{m-1}, \mathcal{J}_{m-1}; x) = \sum_{i=1}^{m-1} \left\{ x^{m-i-1} x^{\sum_{j=i+1}^{m-1} \gamma(j)} D^*(v_i, G_i; x) \right\}, \quad (3.2)$$

in which

$$\gamma(j) = D_{G_j}^*(u_j, v_j).$$

Proof. Since

$$\mathcal{J}_m = \mathcal{J}_{m-1} : G_m,$$

Then applying *Theorem 2.1(2)*, we get (3.1). Thus, we need to find $D^*(v_{m-1}, \mathcal{J}_{m-1}; x)$. Let w be any vertex of \mathcal{J}_{m-1} , that is, $w \in V(G_i)$ for some $i = 1, 2, \dots, m - 1$. Then, from Fig.3.1., we notice that

$$D_{\mathcal{J}_{m-1}}^*(w, v_{m-1}) = x^{D_{\mathcal{J}_{m-1}}^*(w, v_{m-1})} = D_{G_i}^*(w, v_i) + (m - i - 1) + \sum_{j=i+1}^{m-1} D_{G_j}^*(v_j, u_j).$$

The polynomial corresponding to this restricted distance from v_{m-1} in \mathcal{J}_{m-1} is

$$R_i(w; x) = x^{m-i-1} x^{D_{G_i}^*(w, v_i)} x^{\sum_{j=i+1}^{m-1} \gamma(j)},$$

in which

$$\gamma(j) = D_{G_j}^*(u_j, v_j).$$

Therefore, summing for all w in $V(G_i)$, we get

$$\begin{aligned} F_i(x) &= \sum_{w \in V(G_i)} R_i(w; x) = x^{m-i-1} x^{\sum_{j=i+1}^{m-1} \gamma(j)} \sum_{w \in V(G_i)} x^{D_{G_i}(w, v_i)} \\ &= x^{m-i-1} x^{\sum_{j=i+1}^{m-1} \gamma(j)} D_{G_i}^*(v_i, G_i; x). \end{aligned}$$

Now, summing up for $i = 1, 2, \dots, m - 1$, we get the formula (3.2). Hence the proof is completed. ■

Let G_1, G_2, \dots, G_m , be m copies of a connected graph G of order $p \geq 2$ with a pair of fixed distinct vertices u and v as their ends of the new edges introduced in the corresponding copies. Such straight chain of edge-introducing graph is denoted by $\mathcal{J}_m(G)$ which $D^*(v, G; x)$ is the same for each copy of G , and for all j ,

$$\gamma(j) = D_{G_i}(u_i, v_i) = D_G(u, v).$$

Therefore,

$$\sum_{j=i+1}^{m-1} \gamma(j) = (m - i - 1)D_G(u, v).$$

Thus, by *Theorem 3.1*,

$$D^*(v_{m-1}, J_{m-1}(G); x) = \sum_{i=1}^{m-1} \{x^{(m-i-1)} x^{(m-i-1)D_G(u,v)} D^*(v, G; x)\}.$$

Hence, we have the next result:

Theorem 3.2. For $m \geq 2$,

$$D^*(J_m; x) = mD^*(G; x) + xD^*(u, G; x)D^*(v, G; x) \left\{ \sum_{j=1}^{m-1} \sum_{i=1}^{m-j} x^{(m-i-j)(1+D_G^*(u,v))} \right\}. \quad (3.3)$$

Proof. From (3.1) in *Theorem 3.1*, we get for $m \geq 2$,

$$D^*(J_m; x) = D^*(J_{m-1}; x) + D^*(G; x) + xD^*(u, G; x)D^*(v, G; x) \left\{ \sum_{i=1}^{m-1} x^{(m-i-j)(1+D_G^*(u,v))} \right\}.$$

Solving this recurrence relation, we obtain the required formula (3.3). ■

Corollary 3.3. For $m \geq 2$,

$$D^*(J_m(C_6); x) = m(6 + 6x + 3x^3 + 6x^4) + x(1 + 2x + x^3 + 2x^4)^2 \left\{ \sum_{j=1}^{m-1} \sum_{i=1}^{m-j} x^{4(m-i-j)} \right\}. \quad (3.4)$$

in which u and v are diametrical vertices in C_6 .

Proof. From the proof of *Theorem 2.4*, we have $D_{C_6}^*(u, v)$, $D^*(u, C_6; x)$, $D^*(v, C_6; x)$ and $D^*(C_6; x)$. Then, using *Theorem 3.2*, and simplify the result, we obtain (3.4). ■

Corollary 3.4. For $m \geq 2$,

$$D^*(J_m(C_5); x) = m(5 + 5x + 5x^3) + x(1 + 2x + 2x^3)^2 \left\{ \sum_{j=1}^{m-1} \sum_{i=1}^{m-j} x^{4(m-i-j)} \right\}. \quad (3.5)$$

in which u and v are as shown in *Fig.2.3*.

Proof. From the proof of *Theorem 2.5*, we have $D_{C_5}^*(u, v) = 3$, $D^*(u, C_5; x) = D^*(v, C_5; x)$ and $D^*(C_5; x)$. Then, using *Theorem 3.2*, and simplifying the result, we obtain (3.5). ■

Corollary 3.5. For $m \geq 2$,

- (1) $dd^*(J_m(C_6)) = 24m^3 + 24m^2 - 9m$,
- (2) $dd^*(J_m(C_5)) = \frac{50}{3}m^3 + \frac{5}{2}m^2 + \frac{5}{6}m$.

Proof. Take the derivatives of $D^*(J_m(C_6); x)$ and $D^*(J_m(C_5); x)$, given in *Corollaries 3.3 and 3.4* at $x = 1$, and simplifying the results. ■

References

- [1] Abdullah, H. O. and Mohammed-Saleh, G. A., *Detour Hosoya polynomials of some compound graphs*, Raf. J. Comp. Sci. and Maths., Vol. 7, No.1, pp.139-154. (2010).

- [2] Buckley, F. and Harary, F., *Distance in Graphs*, Addison-Wesley, Redwood, California, USA. 1990.
- [3] Chartrand, G. and Lesniak, L., *Graphs and Digraphs, 2nd edition*, Wadsworth and Brooks Cole, California, USA, 1986.
- [4] Chartrand, G., Johns, G. L. and Tian, S., *Detour distance in graphs*, *Annals of Discrete Maths.*, Vol.(55), pp.127-136. (1993).
- [5] Gutman, I., *Some properties of the Wiener polynomial*, *Graph Theory Notes of New York*, Vol.(XXV), pp.13-18. (1993).
- [6] Mohammed-Saleh, G. A., *On the Detour Distance and Detour Polynomials of Graphs*, Ph.D. Thesis, Salahaddin University/Erbil, 2013.