



## On almost and weakly $I_p$ -continuous functions in topological spaces

Laila S. Abdullah\* And Diyar M. Mohammed\*\*

\* Dept. of Mathematics, / Faculty of Science and Science Education/ School of science, University of Sulaimani, Kurdistan Region-Iraq – Email: laylamath33@yahoo.com

\*\* Dept. of Mathematics, Faculty of Physical and Basic Education/ School of Basic Education, University of Sulaimani, Kurdistan Region-Iraq – Email: diyar.mohamed@univsul.edu.iq

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### Abstract

In this paper, we introduce two new classes of continuous functions in topological spaces called almost  $I_p$ -continuous and weakly  $I_p$ -continuous functions. We obtain several characterizations of them and some of their properties. Also, we investigate their relationships with some other types of functions.

**Key Words:** : Preopen set,  $I_p$ -open set,  $I_p$ -continuous function, Almost  $I_p$ -continuous function, Weakly  $I_p$ -continuous function.

### Introduction

Mashhour [8] introduced the concept of preopen sets. Nasef A.A. and Noiri T. [9] introduced a new class of functions called almost precontinuous, while Levine N. [6], defined the weakly precontinuous functions. In this paper almost and weakly  $I_p$ -continuous functions are introduced by using the concept of  $I_p$ -open sets [2].

Throughout this paper by a space  $(X, \tau)$  (or simply  $X$ ) is meant a topological space on which no separation axioms are assumed unless explicitly stated. The closure and the interior of a subset  $A$  of a space  $X$  is denoted by  $Cl(A)$  and  $Int(A)$ , respectively while,  $I_p$ - $Cl(A)$  and  $I_p$ - $Int(A)$  is denoted by the  $I_p$ -closure and  $I_p$ -interior of  $A$  respectively.

A subset  $A$  of a space  $(X, \tau)$  is called semi-open [6] (resp., preopen[8],  $\alpha$ -open [10],  $\beta$ -open[5] and regular open) set if  $A \subset Cl(Int(A))$  (resp.,  $A \subseteq Int(Cl(A))$ ,  $A \subset Int(Cl(Int(A)))$ ,  $A \subset Cl(Int(Cl(A)))$  and  $A = Int(Cl(A))$ ) and  $A$  is said to be a  $\theta$ -open set in  $X$  for each  $x \in A$ , there exists an open set  $U$  in  $X$  such that  $x \in U \subset Cl(U) \subset A$  [12]. The complement of a semi-open (resp., preopen,  $\alpha$ -open,  $\beta$ -closed, regular open and  $\theta$ -open) set is called semi-closed (resp., preclosed,  $\alpha$ -closed, regular closed and  $\theta$ -closed) . Also  $\tau_s$  (resp.,  $\tau_\theta$ ) denotes the family of all regularly open (resp.,  $\theta$ -open) sets in a space  $X$  that form a topology for  $X$ .

## 1. PRELIMINARIES

In this section we recall some basic definitions which are used in this paper.

**Definition 1.1 [2]:** A subset  $A$  of a topological space  $(X, \tau)$  together with the empty set is called an  **$I_p$ -open set** if  $p\text{Int}(A) \neq \emptyset$ . The family of all  $I_p$ -open sets of  $X$  is denoted by  $I_pO(X, \tau)$  or  $I_pO(X)$ .

**Definition 1.2[12]:** A subset  $A$  of a topological space  $(X, \tau)$  is said to be a  **$\theta$ -open ( $\delta$ -open)** set in  $X$  if for each  $x \in A$ , there exists an open set  $U$  in  $X$  such that  $x \in U \subset \text{Cl}(U) \subset A$  (resp.,  $x \in U \subset \text{Int}(\text{Cl}(U)) \subset A$ ). A point  $x \in X$  is said to be the  **$\theta$ -closure ( $\delta$ -closure)** of  $B \subset X$ , if  $B \cap \text{Cl}(U) \neq \emptyset$  (resp.,  $B \cap \text{Int}(\text{Cl}(U)) \neq \emptyset$ ) for each open (resp., open) set  $U$  in  $X$  containing  $x$ . The complement of a  **$\theta$ -open (resp.,  $\delta$ -open)** set in  $X$  is called a  **$\theta$ -closed (resp.,  $\delta$ -closed)** set in  $X$ .

**Definition 1.3 [8]:** A function  $f: X \rightarrow Y$  is called **pre-continuous** (resp., **semi-continuous**) if  $f^{-1}(U)$  is a preopen (resp., semi-open) set in  $X$  for every open (resp., semi-open) subset  $U$  in  $Y$ .

**Definition 1.4:** A function  $f: X \rightarrow Y$  is said to be **almost precontinuous [9]** (resp., **almost continuous [11]**) if the inverse image of each regular open subset of  $Y$  is a preopen (resp., open) subset in  $X$ .

**Definition 1.5 [3]:** A topological space  $(X, \tau)$  is called **hyperconnected** if every non-empty open subset of  $X$  is dense. A hyperconnected space sometimes is called irreducible.

**Theorem 1.6 [2]:** A topological space  $(X, \tau)$  is hyperconnected if and only if every non-empty  $I_p$ -open set in  $X$  is dense.

**Corollary 1.7 [2] :** If a space  $(X, \tau)$  is hyperconnected, then  $I_pO(X) \equiv pO(X)$ , (where  $pO(X)$  is the family of all pre-open sets of a space  $X$ ).

**Definition 1.8 [3]:** A function  $f: (X, \tau) \rightarrow (Y, \sigma)$  is called **regular closed** if the inverse image of each open subset in  $Y$  is a regular closed subset in  $X$ .

**Lemma 1.9[2]:** Every super set of a non-empty  $I_p$ -open set is  $I_p$ -open.

**Lemma 1.10 [2]:** If a non-empty subset  $A$  of a topological space  $(X, \tau)$  is a  $\beta$ -open set in  $X$ , then  $\text{Cl}(A)$  is  $I_p$ -open.

**Definition 1.11[2]:** A function  $f: (X, \tau) \rightarrow (Y, \sigma)$  is called  **$I_p$ -continuous** at  $x \in X$ , if for every open set  $U$  in  $Y$  containing  $f(x)$ , there exists an  $I_p$ -open set  $G$  in  $X$  containing  $x$  such that  $f(G) \subset U$ .  $f$  is said to be  $I_p$ -continuous if it is  $I_p$ -continuous at each point of  $X$ .

**Proposition 1.12[2]:** Every pre-neighbourhood of any point  $x$  in a space  $X$  is an  $I_p$ -open set.

**Lemma 1.13 [1]:** Every regular open subset of a space  $X$  is  $\delta$ -open.

**Proposition 1.14[2]:** A function  $f: (X, \tau) \rightarrow (Y, \sigma)$  is  $I_p$ -continuous if and only if for every open subset  $A$  of  $Y$ ,  $f^{-1}(A)$  is  $I_p$ -open in  $X$ .

## 2. Almost and Weakly $I_p$ -Continuous Functions.

In this section, almost  $I_p$ -continuous and weakly  $I_p$ -continuous functions in topological spaces are defined, several characterizations and some properties of these functions are obtained.

**Definition 2.1:** A function  $f: X \rightarrow Y$  is called **almost  $I_p$ -continuous** at  $x \in X$  if for each open set  $H$  in  $Y$  containing  $f(x)$ , there exists an  $I_p$ -open set  $G$  in  $X$  containing  $x$  such that  $f(G) \subset \text{Int}(\text{Cl}(H))$ .

**Definition 2.2:** A function  $f: X \rightarrow Y$  is called **weakly  $I_p$ -continuous** at  $x \in X$  if for each open set  $H$  in  $Y$  containing  $f(x)$ , there exists an  $I_p$ -open set  $G$  in  $X$  containing  $x$  such that  $f(G) \subset \text{Cl}(H)$ .

It can be stated that  $f: X \rightarrow Y$  is almost (resp., weakly)  $I_p$ -continuous if they are almost (resp., weakly)  $I_p$ -continuous function at each  $x \in X$ .

**Remark 2.3:** Every  $I_p$ -continuous function is almost  $I_p$ -continuous and every almost  $I_p$ -continuous implies weakly  $I_p$ -continuous. But the converse in each case is not true in general as shown in the following examples:

**Example 2.4: (1)** Let  $X = \{1, 2, 3\}$  with  $\tau_1 = \{\varnothing, X, \{1\}, \{2, 3\}\}$  and  $\tau_2 = \{\varnothing, X, \{2\}, \{3\}, \{2, 3\}\}$ . Define a function  $f: (X, \tau_1) \rightarrow (X, \tau_2)$  by:  $f(1)=2, f(2)=3$  and  $f(3)=1$ , then  $f$  is weakly  $I_p$ -continuous but it is neither almost  $I_p$ -continuous nor an  $I_p$ -Continuous function.

**(2)** Let  $X = \{1, 2, 3\}$  and  $\tau_1 = \{\varnothing, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$ ,  $\tau_2 = \{\varnothing, X, \{1\}, \{1, 2\}, \{1, 3\}\}$ . Then define a function  $f: (X, \tau_1) \rightarrow (X, \tau_2)$  by:  $f(1)=3, f(2)=2$  and  $f(3)=1$ , then  $f$  is an almost  $I_p$ -continuous function but not  $I_p$ -continuous.

**Proposition 2.5:** Every almost pre-continuous function (resp., weakly pre-continuous) is an almost  $I_p$ -continuous (resp., weakly  $I_p$ -continuous) function.

**Proof:** The proof is obvious.

However, the converse of Proposition 2.5 is not true in general for both cases as shown in the following example:

**Example 2.6:** Let  $X = \{a, b, c, d\}$  with  $\tau_1 = \{\varnothing, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{b, c, d\}, \{a, b, c\}\}$  and  $\tau_2 = \{\varnothing, X, \{a\}, \{b\}, \{b, c\}, \{a, b\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$ . Let  $f: (X, \tau_1) \rightarrow (X, \tau_2)$  be a function which is defined by  $f(a) = f(b) = f(d) = b$  and  $f(c) = a$ . Then  $f$  is both almost and weakly  $I_p$ -continuous but it is neither almost pre-continuous nor weakly pre-continuous function.

Recall that a topological space  $(X, \tau)$  is said to be **locally indiscrete** if every open subset of  $X$  is closed [7].

In the following proposition we give some conditions which make the converse of (Proposition 2.5) true.

**Proposition 2.7:** Let  $f: X \rightarrow Y$  be a function, where  $X$  is a hyperconnected (or locally indiscrete) space. Then  $f$  is almost pre-continuous (resp., weakly pre-continuous) if and only if  $f$  is an almost  $I_p$ -continuous (resp., weakly  $I_p$ -continuous) function.

**Proof:** Follows from Proposition 2.5 and corollary 1.7

**Theorem 2.8:** If a function  $f: X \rightarrow Y$  is weakly  $I_p$ -continuous, then for each  $x \in X$  and each  $\theta$ -open set  $H$  in  $Y$  containing  $f(x)$ , there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset H$ .

**Proof:** Let  $f$  be weakly  $I_p$ -continuous and let  $x \in X$  and  $H$  be a  $\theta$ -open set in  $Y$  containing  $f(x)$ . Then there exists an open set  $G$  in  $Y$  such that  $f(x) \in G \subset Cl(G) \subset H$ . Since  $f$  is weakly  $I_p$ -continuous, then there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset Cl(G) \subset H$ .

**Corollary 2.9:** If a function  $f: X \rightarrow Y$  is weakly  $I_p$ -continuous, then the inverse image of each  $\theta$ -open set in  $Y$  is an  $I_p$ -open set in  $X$ .

**Proof:** Let  $H$  be a  $\theta$ -open set in  $Y$ . To show  $f^{-1}(H)$  is  $I_p$ -open in  $X$ , let  $f(x) \in H$ . Since  $f$  is weakly  $I_p$ -continuous, then by (Theorem 2.8) there exists an  $I_p$ -open set  $U$  in  $X$  such that  $x \in U$  and  $f(U) \subset H$ . This implies that  $f(x) \in f(U) \subset H$  for all  $x \in X$ . Since  $f(U) \subset H$ , then  $U \subset f^{-1}(f(U)) \subset f^{-1}(H)$ . So  $U \subset f^{-1}(H)$ . Since  $U$  is  $I_p$ -open, then by Lemma 1.9  $f^{-1}(H)$  is an  $I_p$ -open set in  $X$ .

**Corollary 2.10:** If a function  $f: X \rightarrow Y$  is weakly  $I_p$ -continuous, then the inverse image of each  $\theta$ -closed set in  $Y$  is an  $I_p$ -closed set in  $X$ .

**Proof:** Follows from Corollary 2.9, by complementation.

**Corollary 2.11:** Let  $f: X \rightarrow Y$  be a function. If for each  $x \in X$  and each open set  $H$  in  $Y$  containing  $f(x)$ , there exists a  $\beta$ -open set  $U$  in  $X$  containing  $x$  such that  $f(Cl(U)) \subset Int(Cl(H))$  (resp.,  $f(Cl(U)) \subset Cl(H)$ ), then  $f$  is an almost  $I_p$ -continuous (resp., weakly  $I_p$ -continuous) function.

**Proof:** Since  $U$  is  $\beta$ -open, then by (Lemma 1.10)  $Cl(U)$  is  $I_p$ -open. Take  $G = Cl(U)$ , then  $f$  is an almost (resp., weakly)  $I_p$ -continuous function.

**Theorem 2.12:** If a function  $f: (X, \tau) \rightarrow (Y, \sigma)$  is weakly  $I_p$ -continuous, then  $f: (X, \tau) \rightarrow (Y, \sigma_\theta)$  is  $I_p$ -continuous.

**Proof:** Let  $O \in \sigma_\theta$ . Then  $O$  is  $\theta$ -open. Since  $f$  is weakly  $I_p$ -continuous, then by (Corollary 2.9)  $f^{-1}(O)$  is an  $I_p$ -open set in  $X$  and then by (Definition 1.11)  $f$  is  $I_p$ -continuous.

A space  $X$  is called an extremally disconnected, if the closure of each open set in  $X$  is open [7].

**Theorem 2.13:** Let  $f: X \rightarrow Y$  be a function and  $Y$  be an extremally disconnected space. Then  $f$  is weakly  $I_p$ -continuous if and only if the inverse image of each  $\theta$ -open set in  $Y$  is an  $I_p$ -open set in  $X$ .

**Proof:** Let  $f$  be a weakly  $I_p$ -continuous function. Then by (Corollary 2.9) the inverse image of each  $\theta$ -open set in  $Y$  is an  $I_p$ -open set in  $X$ .

Conversely, let the inverse image of each  $\theta$ -open set in  $Y$  be an  $I_p$ -open set in  $X$ . To show that  $f$  is weakly  $I_p$ -continuous, let  $H$  be any open set in  $Y$  containing  $f(x)$ . Then  $\text{Cl}(H)$  is  $\theta$ -open, therefore by the hypothesis  $f^{-1}(\text{Cl}(H))$  is  $I_p$ -open, but  $f(f^{-1}(\text{Cl}(H))) \subset \text{Cl}(H)$ . This implies that  $f$  is weakly  $I_p$ -continuous.

**Theorem 2.14:** Let  $f: X \rightarrow Y$  be a function. If the inverse image of each regular closed set in  $Y$  is an  $I_p$ -open set in  $X$ , then  $f$  is weakly  $I_p$ -continuous.

**Proof:** Let  $H$  be any open subset of  $Y$  containing  $f(x)$ . Then  $\text{Cl}(H)$  is a regular closed set in  $Y$  containing  $f(x)$ . Therefore by the hypothesis  $f^{-1}(\text{Cl}(H))$  is  $I_p$ -open in  $X$  containing  $x$ , and  $f(f^{-1}(\text{Cl}(H))) \subset \text{Cl}(H)$ . Take  $f^{-1}(\text{Cl}(H)) = U$ , then  $U$  is  $I_p$ -open, such that  $f(U) \subset \text{Cl}(H)$ . Thus  $f$  is a weakly  $I_p$ -continuous function.

**Theorem 2.15:** A function  $f: X \rightarrow Y$  is almost  $I_p$ -continuous if and only if for each  $x \in X$  and each  $\delta$ -open set  $H$  in  $Y$  containing  $f(x)$ , there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset H$ .

**Proof:** Let  $f$  be an almost  $I_p$ -continuous function, and let  $x \in X$  and  $H$  be a  $\delta$ -open set in  $Y$  containing  $f(x)$ . Then there exists an open set  $G$  in  $Y$  such that  $f(x) \in G \subset \text{Int}(\text{Cl}(G)) \subset H$ . Since  $f$  is almost  $I_p$ -continuous, then there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset \text{Int}(\text{Cl}(G)) \subset H$ . Thus  $f(U) \subset H$ .

Conversely; let for each  $x \in X$  and each  $\delta$ -open set  $H$  in  $Y$  containing  $f(x)$ , there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset H$ . To show  $f$  is almost  $I_p$ -continuous, let  $x \in X$  and  $G$  be an open set in  $Y$  containing  $f(x)$ , then  $\text{Int}(\text{Cl}(G))$  is  $\delta$ -open set. Therefore, by hypothesis there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset \text{Int}(\text{Cl}(G))$  this implies that  $f$  is an almost  $I_p$ -continuous function.

**Corollary 2.16:** A function  $f: X \rightarrow Y$  is almost  $I_p$ -continuous if and only if the inverse image of each  $\delta$ -open set in  $Y$  is  $I_p$ -open set in  $X$ .

**Proof:** It is followed by Theorem 2.15.

**Corollary 2.17:** A function  $f: X \rightarrow Y$  is almost  $I_p$ -continuous if and only if the inverse image of each  $\delta$ -closed set in  $Y$  is an  $I_p$ -closed set in  $X$ .

**Proof:** Obvious.

**Theorem 2.18:** A function  $f: X \rightarrow Y$  is weakly  $I_p$ -continuous if and only if for each open set  $V$  of  $Y$ ,

$$f^{-1}(V) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(V))).$$

**Proof:** Let  $f: X \rightarrow Y$  be weakly  $I_p$ -continuous, let  $V$  be any open subset of  $Y$  and let  $x \in f^{-1}(V)$ . Then  $f(x) \in V$ . Since  $f$  is weakly  $I_p$ -continuous, then there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset \text{Cl}(V)$ . This implies that  $x \in U \subset f^{-1}(\text{Cl}(V))$ . Therefore,  $x \in I_p\text{-Int}(f^{-1}(\text{Cl}(V)))$ . Thus  $f^{-1}(V) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(V)))$ . Conversely; let  $x \in X$  and  $H$  be any open set in  $Y$  containing  $f(x)$ . Then  $x \in f^{-1}(H)$  and by the hypothesis  $f^{-1}(H) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(H)))$ , this implies that  $x \in f^{-1}(H) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(H))) \subset f^{-1}(\text{Cl}(H))$ , take  $U = I_p\text{-Int}(f^{-1}(\text{Cl}(H)))$ , which is an  $I_p$ -open set in  $X$  containing  $x$  such that  $f(U) \subset \text{Cl}(H)$ . Thus,  $f$  is a weakly  $I_p$ -continuous function.

**Theorem 2.19:** A function  $f: X \rightarrow Y$  is almost  $I_p$ -continuous if and only if for each open set  $V$  of  $Y$ ,

$$f^{-1}(V) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(V)))).$$

**Proof:** Let  $f: X \rightarrow Y$  be almost  $I_p$ -continuous and let  $V$  be any open subset of  $Y$ . To show

$f^{-1}(V) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(V))))$ , let  $x \in f^{-1}(V)$ . Then  $f(x) \in V$ . Since  $f$  is almost  $I_p$ -continuous, then there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset \text{Int}(\text{Cl}(V))$ . This implies that  $x \in U \subset f^{-1}(\text{Int}(\text{Cl}(V)))$ . Therefore,  $x \in I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(V))))$ . Thus  $f^{-1}(V) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(V))))$ .

Conversely; let for each open set  $V$  of  $Y$ ,  $f^{-1}(V) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(V))))$ . To show  $f$  is almost  $I_p$ -continuous, let  $x \in X$  and  $H$  be any open set in  $Y$  containing  $f(x)$ . Then  $x \in f^{-1}(H)$  and by the hypothesis

$f^{-1}(H) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(H))))$ . This implies that  $x \in f^{-1}(H) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(H)))) \subset f^{-1}(\text{Int}(\text{Cl}(H)))$ , take

$U = I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(H))))$  which is an  $I_p$ -open set in  $X$  containing  $x$  such that  $f(U) \subset \text{Int}(\text{Cl}(H))$ . Thus  $f$  is an almost  $I_p$ -continuous function.

**Corollary 2.20:** A function  $f: X \rightarrow Y$  is a weakly  $I_p$ -continuous (resp., almost  $I_p$ -continuous) function if and only if for each  $B \in \beta$ , where  $\beta$  is a basis for a topology on  $Y$ ,  $f^{-1}(B) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(B)))$  (resp.,  $f^{-1}(B) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(B))))$ ).

**Proof:** Let  $f: X \rightarrow Y$  be a weakly  $I_p$ -continuous (resp. almost  $I_p$ -continuous) function and let  $B \in \beta$ , then  $B$  is an open subset of  $Y$ , thus by Theorems 2.18 (resp., Theorem 2.19)  $f^{-1}(B) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(B)))$  (resp.,  $f^{-1}(B) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(B))))$ ).

Conversely; let for each  $B \in \beta$ ,  $f^{-1}(B) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(B)))$  (resp.,  $f^{-1}(B) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(B))))$ ). Let  $H$  be any open set in  $Y$  containing  $f(x)$ , then  $H = \bigcup_{i \in I} B_i$ ,  $B_i \in \beta$  for each  $i \in I$ , by hypothesis for each  $i \in I$ ,  $f^{-1}(B_i) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(B_i)))$  (resp.,  $f^{-1}(B_i) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(B_i))))$ ). Therefore,  $\bigcup_{i \in I} f^{-1}(B_i) \subset \bigcup_{i \in I} I_p\text{-Int}(f^{-1}(\text{Cl}(B_i)))$  (resp.,  $\bigcup_{i \in I} f^{-1}(B_i) \subset \bigcup_{i \in I} I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(B_i))))$ ), then  $f^{-1}(\bigcup_{i \in I} B_i) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(\bigcup_{i \in I} B_i)))$  (resp.,  $f^{-1}(\bigcup_{i \in I} B_i) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(\bigcup_{i \in I} B_i))))$ ).

$f^{-1}(\bigcup_{i \in I} B_i) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(\bigcup_{i \in I} B_i))))$ . Therefore,  $f^{-1}(H) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(H)))$  (resp.,  $f^{-1}(H) \subset I_p\text{-Int}(f^{-1}(\text{Int}(\text{Cl}(H))))$ ). Thus, by Theorem 2.18, (resp., Theorem 2.19)  $f$  is a weakly  $I_p$ -continuous (resp., an almost  $I_p$ -continuous) function.

**Corollary 2.21:** A function  $f: X \rightarrow Y$  is almost  $I_p$ -continuous if and only if the inverse image of each regular open set in  $Y$  is an  $I_p$ -open set in  $X$ .

**Proof:** Since every regular open is  $\delta$ -open, so the result is followed by Corollary 2.16.

**Corollary 2.22:** A function  $f: X \rightarrow Y$  is almost  $I_p$ -continuous if and only if the inverse image of each regular closed set in  $Y$  is an  $I_p$ -closed set in  $X$ .

**Proof:** Clear.

The following result is another characterization of almost  $I_p$ -continuous functions.

**Theorem 2.23:** For the function  $f: X \rightarrow Y$ , the following statements are equivalent:

- 1)  $f$  is an almost  $I_p$  -continuous function.
- 2)  $f(I_p\text{-Cl}(A)) \subset Cl_\delta f(A)$ , for each subset  $A$  of  $X$ .
- 3)  $I_p\text{-Cl}(f^{-1}(B)) \subset f^{-1}(Cl_\delta(B))$ , for each subset  $B$  of  $Y$ .
- 4)  $f^{-1}(\text{Int}_\delta(B)) \subset I_p\text{-Int}(f^{-1}(B))$ , for each subset  $B$  of  $Y$ .
- 5)  $I_p\text{-Cl}(f^{-1}(B)) \subset f^{-1}(\text{Cl}(B))$ , for each semi-open set  $B$  in  $Y$ .

**Proof:** (1)  $\Rightarrow$  (2) Let  $A$  be any subset of  $X$ . To show  $f(I_p\text{-Cl}(A)) \subset Cl_\delta f(A)$ , suppose that  $f(I_p\text{-Cl}(A)) \not\subset Cl_\delta f(A)$ . This implies that there exists  $y \in f(I_p\text{-Cl}(A))$  such that  $y \notin Cl_\delta(f(A))$ , then there exists an open set  $H$  in  $Y$  containing  $y$  such that  $\text{Int}(\text{Cl}(H)) \cap f(A) = \emptyset$ . If  $f^{-1}(y) = \emptyset$ , then there is nothing to prove, if  $f^{-1}(y) \neq \emptyset$ , suppose that  $x$  is any arbitrary point of  $f^{-1}(y)$ . Therefore,  $f(x) \in H$ , but  $H$  is an open subset of  $Y$ , then by (1) there exists an  $I_p$  -open set  $U$  of  $X$  containing  $x$  such that  $f(U) \subset \text{Int}(\text{Cl}(H))$ , so  $f(U) \cap f(A) = \emptyset$ . This implies that  $f(U \cap A) = \emptyset$ , then  $U \cap A = \emptyset$ . Thus  $x \notin I_p(\text{Cl}(A))$ . Therefore,  $y \notin f(I_p(\text{Cl}(A)))$  which is a contradiction.

Hence  $f(I_p\text{-Cl}(A)) \subset Cl_\delta(f(A))$ .

(2)  $\Rightarrow$  (3) Let  $B$  be any subset of  $Y$ . Therefore, by (2)  $f(I_p\text{-Cl}(f^{-1}(B))) \subset Cl_\delta(f(f^{-1}(B))) \subset Cl_\delta(B)$ . This implies that  $f^{-1}(f(I_p\text{-Cl}(f^{-1}(B)))) \subset f^{-1}(Cl_\delta(B))$ , then  $I_p\text{-Cl}(f^{-1}(B)) \subset f^{-1}(Cl_\delta(B))$ .

(3)  $\Rightarrow$  (4) Let  $B$  be any subset of  $Y$ . Then  $Y \setminus B$  is a subset of  $Y$ . Therefore, by (3)  $I_p\text{-Cl}(f^{-1}(Y \setminus B)) \subset f^{-1}(Cl_\delta(Y \setminus B))$ . This implies that  $I_p\text{-Cl}(X \setminus f^{-1}(B)) \subset f^{-1}(Y \setminus \text{Int}_\delta(B))$ , and so  $X \setminus I_p\text{-Int}(f^{-1}(B)) \subset X \setminus f^{-1}(\text{Int}_\delta(B))$ . Hence  $f^{-1}(\text{Int}_\delta(B)) \subset I_p\text{-Int}(f^{-1}(B))$ .

(4)  $\Rightarrow$  (5) Let  $B$  be any semi-open subset of  $Y$ . Then  $\text{Cl}(B)$  is regular closed in  $Y$ . Therefore,  $Y \setminus \text{Cl}(B)$  is regular open in  $Y$ . Thus by (Lemma 1.12)  $Y \setminus \text{Cl}(B)$  is  $\delta$ -open in  $Y$ , then by (4)  $f^{-1}(Y \setminus \text{Cl}(B)) \subset I_p\text{-Int}(f^{-1}(Y \setminus \text{Cl}(B)))$ . This implies that  $f^{-1}(Y \setminus \text{Cl}(B))$  is an  $I_p$ -open set, and then  $X \setminus f^{-1}(\text{Cl}(B))$  is an  $I_p$ -open set in  $X$ , that is  $f^{-1}(\text{Cl}(B))$  is  $I_p$ -closed. But  $f^{-1}(B) \subset f^{-1}(\text{Cl}(B))$ . This implies that  $I_p\text{-Cl}(f^{-1}(B)) \subset f^{-1}(\text{Cl}(B))$ .

(5)  $\Rightarrow$  (1) Let  $B$  be any regular open subset of  $Y$ , then  $Y \setminus B$  is a regular closed set in  $Y$ . This implies that  $Y \setminus B$  is a semi-open subset of  $Y$ . Therefore, by (5)  $I_p\text{-Cl}(f^{-1}(Y \setminus B)) \subset f^{-1}(\text{Cl}(Y \setminus B)) = f^{-1}(Y \setminus B)$ , hence  $f^{-1}(Y \setminus B)$  is  $I_p$  -closed in  $X$ . That is,  $X \setminus f^{-1}(B)$  is  $I_p$ -closed. Therefore,  $f^{-1}(B)$  is an  $I_p$ -open set. Hence by Corollary 2.21,  $f$  is almost  $I_p$ -continuous.

The following theorem is a characterization of weakly  $I_p$ -continuous functions.

**Theorem 2.24:** For a function  $f: X \rightarrow Y$ , the following statements are equivalent:

1.  $f$  is a weakly  $I_p$ -continuous.
2.  $f(I_p\text{-Cl}(A)) \subset Cl_0(f(A))$ , for each subset  $A$  of  $X$ .
3.  $I_p\text{-Cl}(f^{-1}(B)) \subset f^{-1}(Cl_0(B))$ , for each subset  $B$  of  $Y$ .
4.  $f^{-1}(Int_0(B)) \subset I_p\text{-Int}(f^{-1}(B))$ , for each subset  $B$  of  $Y$ .
5.  $I_p\text{-Cl}(f^{-1}(B)) \subset f^{-1}(Cl(B))$ , for each open set  $B$  in  $Y$ .
6.  $f^{-1}(Int(Cl(H))) \subset I_p\text{-Int}(f^{-1}(Cl(H)))$ , for each open set  $H$  in  $Y$ .
7.  $I_p\text{-Cl}f^{-1}(Int(F)) \subset f^{-1}(Cl(Int(F)))$ , for each closed set  $F$  in  $Y$ .

**Proof:** (1)  $\Rightarrow$  (2) Let  $A$  be any subset of  $X$ , we suppose that  $f(I_p\text{-Cl}(A)) \not\subset Cl_0(f(A))$ , this implies that there exists  $y \in f(I_p\text{-Cl}(A))$  such that  $y \notin Cl_0(f(A))$ . Then there exists an open set  $H$  in  $Y$  containing  $y$  such that  $Cl(H) \cap f(A) = \emptyset$ . If  $f^{-1}(y) = \emptyset$ , then there is nothing to prove. Suppose that  $x$  is any arbitrary point of  $f^{-1}(y)$ , therefore;  $f(x) \in H$ , but  $H$  is an open subset of  $Y$ , then by hypothesis there exists an  $I_p$ -open set  $U$  of  $X$  containing  $x$  such that  $f(U) \subset Cl(H)$ . Then  $f(U) \cap f(A) = \emptyset$ , which implies that  $f(U \cap A) = \emptyset$ , then  $U \cap A = \emptyset$ . Thus  $x \notin I_p\text{-Cl}(A)$ , therefore  $y \notin f(I_p\text{-Cl}(A))$  which is a contradiction. Hence  $f(I_p\text{-Cl}(A)) \subset Cl_0(f(A))$ .

(2)  $\Rightarrow$  (3) Let  $B$  be any subset of  $Y$ . Then  $f^{-1}(B) \subset X$ , therefore by (2)  $f(I_p\text{-Cl}(f^{-1}(B))) \subset Cl_0(f(f^{-1}(B))) \subset Cl_0(B)$ , then  $f^{-1}(f(I_p\text{-Cl}(f^{-1}(B)))) \subset f^{-1}(Cl_0(B))$ . Hence  $I_p\text{-Cl}(f^{-1}(B)) \subset f^{-1}(Cl_0(B))$ .

(3)  $\Rightarrow$  (4) Let  $B$  be any subset of  $Y$ . Then  $Y \setminus B$  is also a subset of  $Y$ , therefore by (3)

$I_p\text{-Cl}(f^{-1}(Y \setminus B)) \subset f^{-1}(Cl_0(Y \setminus B))$ , so  $I_p\text{-Cl}(X \setminus f^{-1}(B)) \subset f^{-1}(Y \setminus Int_0(B))$ . This implies that

$X \setminus I_p\text{-Int}(f^{-1}(B)) \subset X \setminus f^{-1}(Int_0(B))$ , hence  $f^{-1}(Int_0(B)) \subset I_p\text{-Int}(f^{-1}(B))$ .

(4)  $\Rightarrow$  (5) Let  $H$  be any open subset of  $Y$ . Then  $Y \setminus H$  is closed in  $Y$ . Since  $H$  is an open set in  $Y$ , then  $Cl_0(H) = Cl(H)$ , and by (4)  $f^{-1}(Int_0(Y \setminus H)) \subset I_p\text{-Int}(f^{-1}(Y \setminus H))$ . This implies that  $f^{-1}(Y \setminus Cl_0(H)) \subset$

$I_p\text{-Int}(X \setminus f^{-1}(H))$ , that is  $X \setminus f^{-1}(Cl(H)) \subset X \setminus I_p\text{-Cl}(f^{-1}(H))$ . Thus  $I_p\text{-Cl}(f^{-1}(H)) \subset f^{-1}(Cl(H))$ .

(5)  $\Rightarrow$  (6) Let  $H$  be any open subset of  $Y$ . Then  $Y \setminus Cl(H)$  is open in  $Y$ , so by (5)  $I_p\text{-Cl}(f^{-1}(Y \setminus Cl(H))) \subset$

$f^{-1}(Cl(Y \setminus Cl(H)))$ . Therefore,  $I_p\text{-Cl}(X \setminus f^{-1}(Cl(H))) \subset f^{-1}(Y \setminus Int(Cl(H)))$ , hence  $X \setminus I_p\text{-Int}(f^{-1}(Cl(H))) \subset$

$X \setminus f^{-1}(Int(Cl(H)))$ . Thus  $f^{-1}(Int(Cl(H))) \subset I_p\text{-Int}(f^{-1}(Cl(H)))$ .

(6)  $\Rightarrow$  (7) Let  $F$  be any closed subset of  $Y$ . Then  $Y \setminus F$  is open. By (6)  $f^{-1}(\text{Int}(\text{Cl}(Y \setminus F))) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(Y \setminus F)))$ . Therefore,  $f^{-1}(\text{Int}(Y \setminus \text{Int}(F))) \subset I_p\text{-Int}(f^{-1}(Y \setminus \text{Int}(F)))$ . This implies that  $f^{-1}(Y \setminus \text{Cl}(\text{Int}(F))) \subset I_p\text{-Int}(X \setminus f^{-1}(\text{Int}(F)))$ , then  $X \setminus f^{-1}(\text{Cl}(\text{Int}(F))) \subset X \setminus I_p\text{-Cl}(f^{-1}(\text{Int}(F)))$ . Thus,  $I_p\text{-Cl}(f^{-1}(\text{Int}(F))) \subset f^{-1}(\text{Cl}(\text{Int}(F)))$ .

(7)  $\Rightarrow$  (6) Let  $H$  be an open subset in  $Y$ , then  $Y \setminus H$  is a closed set, and by (7)  $I_p\text{-Cl}(f^{-1}(\text{Int}(Y \setminus H))) \subset f^{-1}(\text{Cl}(\text{Int}(Y \setminus H)))$ . Then  $I_p\text{-Cl}(f^{-1}(Y \setminus \text{Cl}(H))) \subset f^{-1}(\text{Cl}(Y \setminus \text{Cl}(H)))$  and so  $I_p\text{-Cl}(X \setminus f^{-1}(\text{Cl}(H))) \subset f^{-1}(Y \setminus \text{Int}(\text{Cl}(H)))$ , then  $X \setminus I_p\text{-Int}(f^{-1}(\text{Cl}(H))) \subset X \setminus f^{-1}(\text{Int}(\text{Cl}(H)))$ . Thus  $f^{-1}(\text{Int}(\text{Cl}(H))) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(H)))$ .

(6)  $\Rightarrow$  (1) Let  $H$  be any open subset of  $Y$  containing  $f(x)$ , where  $x \in X$ , then  $x \in f^{-1}(H)$ . Since  $H$  is open, then  $f^{-1}(H) \subset f^{-1}(\text{Int}(\text{Cl}(H)))$ , thus by (6)  $x \in f^{-1}(\text{Int}(\text{Cl}(H))) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(H)))$ . This implies that  $x \in I_p\text{-Int}(f^{-1}(\text{Cl}(H)))$ . Therefore, there exists an  $I_p$ -open set  $A$  of  $X$  such that  $x \in A \subset f^{-1}(\text{Cl}(H))$ , then  $f(A) \subset f(f^{-1}(\text{Cl}(H))) \subset \text{Cl}(H)$ . That is  $f(A) \subset \text{Cl}(H)$ . Hence  $f$  is a weakly  $I_p$ -continuous function.

**Corollary 2.25:** A function  $f: (X, \tau) \rightarrow (Y, \sigma)$  is almost  $I_p$ -continuous if and only if

$f: (X, \tau) \rightarrow (Y, \sigma_s)$  is  $I_p$ -continuous.

**Proof:** It is a direct consequence of Proposition 1.14 and Corollary 2.16.

**Theorem 2.26:** For an extremally disconnected space  $Y$ , a function  $f: X \rightarrow Y$  is almost  $I_p$ -continuous if and only if  $f$  is weakly  $I_p$ -continuous.

**Proof:** If  $f$  is almost  $I_p$ -continuous then  $f$  is weakly  $I_p$ -continuous.

Conversely, let  $f$  be weakly  $I_p$ -continuous and let  $H$  be any open subset of  $Y$  containing

$f(x)$ , where  $x$  in  $X$ . Then there exists an  $I_p$ -open set  $U$  in  $X$  containing  $x$  such that  $f(U) \subset \text{Cl}(H)$ . Since  $Y$  is extremally disconnected, then  $\text{Cl}(H) = \text{Int}(\text{Cl}(H))$ , that is  $f(U) \subset \text{Cl}(H) = \text{Int}(\text{Cl}(H))$ . Thus  $f$  is an almost  $I_p$ -continuous function.

**Corollary 2.27:** Let  $f: X \rightarrow Y$  be a function and  $\beta$  be a basis for  $\sigma_\theta$  in  $Y$ . If  $f$  is weakly  $I_p$ -continuous, then

$f^{-1}(B)$  is an  $I_p$ -open subset of  $X$  for each  $B \in \beta$ .

**Proof:** Assume that  $f$  is weakly  $I_p$ -continuous and let  $B \in \beta$ , and then  $B$  is a  $\theta$ -open subset of  $Y$ . Since  $f$  is weakly  $I_p$ -continuous, then by Corollary 2.9,  $f^{-1}(B)$  is an  $I_p$ -open subset of  $X$ .

**Corollary 2.28:** Let  $f: X \rightarrow Y$  be a function and  $\beta$  be a basis for  $\tau_s$  in  $Y$ . Then  $f$  is almost  $I_p$ -continuous if and only if for each  $B \in \beta$ ,  $f^{-1}(B)$  is an  $I_p$ -open subset of  $X$ .

**Proof:** Since the base of  $\tau_s$  is the set of all regular open subset of  $Y$ , so the result is followed by Corollary 2.21.

**Proposition 2.29:** If  $Y$  is a hyperconnected space, then every function  $f: X \rightarrow Y$  is almost  $I_p$ -continuous.

**Proof:** Let  $Y$  be a hyperconnected space and  $H$  be any open subset of  $Y$  containing  $f(x)$ , then  $Cl(H)=Y$ , this implies that  $Int(Cl(H))=Y$  and for every  $I_p$ -open set  $G$  in  $X$  containing  $x$ ,  $f(G) \subset Y = Int(Cl(H))$ , thus  $f$  is an almost  $I_p$ -continuous function.

**Corollary 2.30:** If  $Y$  is a hyperconnected space, then every function  $f: X \rightarrow Y$  is weakly  $I_p$ -continuous.

The following theorem is a characterization of weakly  $I_p$ -continuous functions.

**Theorem 2.31:** For a function  $f: X \rightarrow Y$  the following statements are equivalent:

1.  $f$  is a weakly  $I_p$ -continuous function.
2. For each  $B \in \beta$ , where  $\beta$  is a basis for a topology on  $Y$ ,  $f^{-1}(B) \subset I_p - Int(f^{-1}(Cl(B)))$ .
3. For each open subset  $V$  of  $Y$ ,  $f^{-1}(V) \subset I_p - Int(f^{-1}(Cl(V)))$ .
4. For each open subset  $V$  of  $Y$ ,  $I_p - Cl(f^{-1}(V)) \subset f^{-1}(Cl(V))$ .

**Proof:** (1)  $\Rightarrow$  (2) Let  $B \in \beta$ ; where  $\beta$  a basis for a topology on  $Y$ . Then by Theorem 2.18  $f^{-1}(B) \subset I_p - Int(f^{-1}(Cl(B)))$ .

(2)  $\Rightarrow$  (3) Let  $V$  be an open subset in  $Y$ . If  $f^{-1}(V) = \varnothing$ , then there is nothing to prove. If  $f^{-1}(V) \neq \varnothing$ , then we take  $x$  arbitrary in it, so  $f(x) \in V$ . Since  $\beta$  a base of  $Y$ , then there is a  $B \in \beta$  such that  $f(x) \in B \subset V$ . Thus

$x \in f^{-1}(B)$  and by (2)  $f^{-1}(B) \subset I_p - Int(f^{-1}(Cl(B)))$ . Since  $B \subset V$ , so  $x \in I_p - Int(f^{-1}(Cl(V)))$ . Hence

$f^{-1}(V) \subset I_p - Int(f^{-1}(Cl(V)))$ .

(3)  $\Rightarrow$  (4) Let  $V$  be an open subset of  $Y$ . Since  $Y \setminus \text{Cl}(V)$  is open in  $Y$ , then by (3)  $f^{-1}(Y \setminus \text{Cl}(V))$

$\subset I_p\text{-Int}(f^{-1}(\text{Cl}(Y \setminus \text{Cl}(V))))$  so  $X \setminus f^{-1}(\text{Cl}(V)) \subset I_p\text{-Int}(f^{-1}(Y \setminus \text{Int}(\text{Cl}(V))))$ , hence  $X \setminus f^{-1}(\text{Cl}(V))$

$\subset X \setminus I_p\text{-Cl}(f^{-1}(\text{Int}(\text{Cl}(V)))) \subset X \setminus (I_p\text{-Cl}(f^{-1}(V)))$  {since  $V$  is open}. Thus  $I_p\text{-Cl}(f^{-1}(V)) \subset f^{-1}(\text{Cl}(V))$ .

(4)  $\Rightarrow$  (1) Let  $V$  be an open subset of  $Y$  containing  $f(x)$  and let  $W = Y \setminus \text{Cl}(V)$ . Then by (4)  $I_p\text{-Cl}(f^{-1}(W))$

$\subset f^{-1}(\text{Cl}(W))$ , so  $I_p\text{-Cl}(f^{-1}(Y \setminus \text{Cl}(V))) \subset f^{-1}(\text{Cl}(Y \setminus \text{Cl}(V)))$ , hence  $X \setminus I_p\text{-Int}(f^{-1}(\text{Cl}(V))) \subset X \setminus f^{-1}(\text{Int}(\text{Cl}(V)))$

$\subset X \setminus f^{-1}(V)$  {since  $V$  is open in  $Y$ }. Thus  $f^{-1}(V) \subset I_p\text{-Int}(f^{-1}(\text{Cl}(V)))$ . So by Theorem 2.18  $f$  is a weakly  $I_p$ -continuous function.  $\hat{\quad}$

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