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Local Fractional Variational Iteration Transform Method for Solving Couple Helmholtz Equations within Local Fractional Operator

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Abstract

In this paper, we investigate the solution of Helmholtz and coupled Helmholtz equations in two dimensional case, involving local fractional variation iteration method and Yang-Laplace transform which is called local fractional Variational iteration transform method (LFVITM). This method has Lagrange multiplier equal to -1, which makes the calculations more easily .The obtained result shows that the proposed method is efficient and accurate.

Introduction

The Helmholtz equation often arises in physical phenomena and engineering applications involving partial differential equations such as heat conduction, acoustic radiation, water wave propagation and even in biology. Kreb and Roach **[1]** discussed the transmission problems for the Helmholtz equation. Kleinman and Roach **[2]** studied the boundary integral equations for the three-dimensional Helmholtz equation. Karageorghis **[3]** presented the eigenvalues of the Helmholtz equation. Fu and Mura **[4]** suggested the volume integrals of the Helmholtz equation. Samuel and Thomas **[5]** proposed the fractional Helmholtz equation.

There are many analytical and numerical methods used to solve local fractional partial differential Equations such as, local fractional function decomposition method **[6,7]**, local fractional Adomian decomposition method **[7,8]**, local fractional series expansion method **[9,10]**, local fractional Laplace transform method **[11,12]**, local fractional Fourier series method **[13]**, local fractional Laplace decomposition method **[14,15]**, local fractional Laplace variational iteration method **[16,17,18]**, and another methods. In this paper, our aim is to present the coupling method of local fractional Laplace transform and variational iteration method, which is called as the local fractional variational iteration transform method, and to used it to solve the Helmholtz and coupled Helmholtz equations within local fractional operator.

Analysis of the local Fractional Variational Iteration Method

Let us consider the following partial differential equation within local fractional derivative operator:

$$L_\alpha u(x,y) + R_\alpha u(x,y) = g(x,y),\ 0<\alpha\le 1, \tag{2.1}$$

where $L_\alpha = \frac{\partial^{n\alpha}}{\partial x^{n\alpha}},\ n\in N$ is the linear local fractional derivative operator, R_α denotes a lower order local fractional derivative operator, and $g(x,y)$ is the nondifferentiable source term.

Local fractional variational iteration algorithm **[10,11]** reads as

$$u_{n+1}(x,y) = u_n(x,y) + {}_0I_x^{(\alpha)}\left\{\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left(L_\alpha u_n(\xi,y) + R_\alpha u_n(\xi,y) - g(\xi,y)\right)\right\}, \tag{2.2}$$

where $\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}$ is a general fractal Lagrange's multiplier.

Therefore, a local fractional correction functional was structured as follows:

$$u_{n+1}(x,y) = u_n(x,y) + {}_0I_x^{(\alpha)}\left\{\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}\left(L_\alpha u_n(\xi,y) + R_\alpha \tilde{u}_n(\xi,y) - g(\xi,y)\right)\right\}, \tag{2.3}$$

where $\tilde{u}_n$ is considered as a restricted local fractional variation. That is, $\delta^\alpha\, \tilde{u}_n = 0$.

After the fractal Lagrangian multiplier is determined, for $n\ge 0$,the successive approximations $u_{n+1}(x,y)$ of the solution $u(x,y)$ can be readily given by using any selective local fractional function $u_0(x,y)$.

For $n=2$, we have

$$\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} = \frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}, \tag{2.4}$$

so that iteration is expressed as

$$u_{n+1}(x,y) = u_n(x,y) + {}_0I_x^{(\alpha)}\left\{\frac{(\xi-x)^\alpha}{\Gamma(1+\alpha)}\left(L_\alpha u_n(\xi,y) + R_\alpha u_n(\xi,y) - g(\xi,y)\right)\right\}. \tag{2.5}$$

Finally, we obtain the solution

$$u(x,y) = \lim_{n\to\infty} u_n(x,y). \tag{2.6}$$

Local Fractional Variational Iteration Transform Method (LFVITM).

Applying the Yang-Laplace transform (denoted in this paper by $Ł_\alpha$) on both sides of (2.1), we get

$$Ł_\alpha\{L_\alpha u(x,y)\} + Ł_\alpha\{R_\alpha u(x,y)\} = Ł_\alpha\{g(x,y)\}. \tag{3.1}$$

Using the property of the Yang-Laplace transform, we have

$$s^{n\alpha}Ł_\alpha\{u(x,y)\} - s^{(n-1)\alpha}u(0,y) - s^{(n-2)\alpha}u_x^{(\alpha)}(0,y) - \cdots - u_x^{((n-1)\alpha)}(0,y) = Ł_\alpha\{g(x,y) - R_\alpha u(x,y)\}, \tag{3.2}$$

or equivalently

$$Ł_\alpha\{u(x,y)\} = \frac{1}{s^\alpha}u(0,y) + \frac{1}{s^{2\alpha}}u_x^{(\alpha)}(0,y) + \cdots + \frac{1}{s^{n\alpha}}u_x^{((n-1)\alpha)}(0,y) + \frac{1}{s^{n\alpha}}Ł_\alpha\{g(x,t) - R_\alpha u(x,y)\}. \tag{3.3}$$

Operating with the Yang-Laplace inverse on both sides of (3.3) gives

$$u(x,y)=u(0,y)+\frac{x^{\alpha}}{\Gamma(1+\alpha)}u_x^{(\alpha)}(0,y)+\cdots+\frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}u_x^{((n-1)\alpha)}(0,y)+ Ł_{\alpha}^{-1}\left(\frac{1}{s^{n\alpha}}Ł_{\alpha}\{g(x,y)-R_{\alpha}u(x,y)\}\right). \tag{3.4}$$

Derivative both side (3.4) with respect to x, we have

$$\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}}=\frac{\partial^{\alpha}}{\partial x^{\alpha}}Ł_{\alpha}^{-1}\left(\frac{1}{s^{n\alpha}}Ł_{\alpha}\{g(x,y)-R_{\alpha}u(x,y)\}\right)+ u_x^{(\alpha)}(0,y)+\cdots+\frac{x^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}u_x^{((n-1)\alpha)}(0,y). \tag{3.5}$$

We now structure the correctional local fractional function in the form

$$u_{m+1}(x,y)=u_m(x,y)+\frac{1}{\Gamma(1+\alpha)}\times \int_0^x\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)}\left(\begin{array}{l}\frac{\partial^{\alpha}u_m(\xi,y)}{\partial\xi^{\alpha}}-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}Ł_{\alpha}^{-1}\left(\frac{1}{s^{n\alpha}}Ł_{\alpha}\{g(\xi,y)-R_{\alpha}u_m(\xi,y)\}\right)-\\ (u_m)_{\xi}^{(\alpha)}(0,y)-\cdots-\frac{\xi^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}(u_m)_{\xi}^{((n-1)\alpha)}(0,y)\end{array}\right)(d\xi)^{\alpha}$$

(3.6)

Making the local fractional variation , we get

$$\delta^{\alpha}u_{m+1}(x,y)=\delta^{\alpha}u_m(x,y)+\delta^{\alpha}\frac{1}{\Gamma(1+\alpha)}\times \int_0^x\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)}\left(\begin{array}{l}\frac{\partial^{\alpha}u_m(\xi,y)}{\partial\xi^{\alpha}}-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}Ł_{\alpha}^{-1}\left(\frac{1}{s^{n\alpha}}Ł_{\alpha}\{g(\xi,y)-R_{\alpha}u_m(\xi,y)\}\right)-\\ (u_m)_{\xi}^{(\alpha)}(0,y)-\cdots-\frac{\xi^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}(u_m)_{\xi}^{((n-1)\alpha)}(0,y)\end{array}\right)(d\xi)^{\alpha}$$

(3.7)

The extremum condition of $u_{n+1}(x,y)$ is given by [11]

$$\delta^{\alpha}u_{m+1}(x,y)=0.$$

(3.8)

In view of (3.8), we have the following stationary conditions:

$$1+\left.\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)}\right|_{\xi=x}=0,\qquad \left.\left(\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)}\right)^{(\alpha)}\right|_{\xi=x}=0\ . \tag{3.9}$$

This is turn gives

$$\frac{\lambda(\xi)^{\alpha}}{\Gamma(1+\alpha)}=-1. \tag{3.10}$$

Substituting (3.10) into (3.6), we obtained

$$u_{m+1}(x,y)=u_m(x,y)-\frac{1}{\Gamma(1+\alpha)}\times \int_0^x \begin{pmatrix} \frac{\partial^\alpha u_m(\xi,y)}{\partial \xi^\alpha}-\frac{\partial^\alpha}{\partial \xi^\alpha}Ł_\alpha^{-1}\left(\frac{1}{s^{n\alpha}}Ł_\alpha\{g(\xi,y)-R_\alpha u_m(\xi,y)\}\right)- \\ (u_m)_\xi^{(\alpha)}(0,y)-\cdots-\frac{\xi^{(n-2)\alpha}}{\Gamma(1+(n-2)\alpha)}(u_m)_\xi^{((n-1)\alpha)}(0,y) \end{pmatrix}(d\xi)^\alpha . \tag{3.11}$$

Finally, the solution $u(x,y)$ is given by

$$u(x,y)=\lim_{m\to\infty}u_m(x,y). \tag{3.12}$$

Illustrate Examples

Example 1. Let us consider the following Helmholtz equation with local fractional derivative operator in the form

$$\frac{\partial^{2\alpha}H(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}H(x,y)}{\partial y^{2\alpha}}-H(x,y)=0, \tag{4.1}$$

subject to the initial value

$$H(0,y)=0,\ \frac{\partial^\alpha H(0,y)}{\partial x^\alpha}=\cosh_\alpha(y^\alpha). \tag{4.2}$$

In view of (3.11) and (4.1) the local fractional iteration algorithm can be written as follows:

$$H_{m+1}(x,y)=H_m(x,y) -\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_m(\xi,y)}{\partial\xi^\alpha}-\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_m(\xi,y)-\frac{\partial^{2\alpha}H_m(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-H_m^{(\alpha)}(0,y)\right)(d\xi)^\alpha \tag{4.3}$$

We can use the initial conditions to select $H_0(x,y)=\frac{x^\alpha}{\Gamma(1+\alpha)}\cosh_\alpha(y^\alpha)$. Using this selection into the correction functional (4.3) gives the following successive approximations

$$H_0(x,y)=\frac{x^\alpha}{\Gamma(1+\alpha)}\cosh_\alpha(y^\alpha),$$

$$H_1(x,y)=H_0(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_0(\xi,y)}{\partial\xi^\alpha}-\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_0(\xi,y)-\frac{\partial^{2\alpha}H_0(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-H_0^{(\alpha)}(0,y)\right)(d\xi)^\alpha$$

$$=\frac{x^\alpha}{\Gamma(1+\alpha)}\cosh_\alpha(y^\alpha) -\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\cosh_\alpha(y^\alpha)-\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{\frac{\xi^\alpha}{\Gamma(1+\alpha)}\cosh_\alpha(y^\alpha)-\frac{\xi^\alpha}{\Gamma(1+\alpha)}\cosh_\alpha(y^\alpha)\right\}\right)\right)-\cosh_\alpha(y^\alpha)\right)(d\xi)^\alpha$$

$$=\frac{x^\alpha}{\Gamma(1+\alpha)}\cosh_\alpha(y^\alpha),$$

$$H_2(x,y)=H_1(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_1(\xi,y)}{\partial\xi^\alpha}-\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_1(\xi,y)-\frac{\partial^{2\alpha}H_1(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-H_1^{(\alpha)}(0,y)\right)(d\xi)^\alpha$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha}),$$

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$$H_m(x,y)=H_{m-1}(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}H_{m-1}(\xi,y)}{\partial\xi^{\alpha}}-\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{H_{m-1}(\xi,y)-\frac{\partial^{2\alpha}H_{m-1}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-H_{m-1}^{(\alpha)}(0,y)\right)(d\xi)^{\alpha}$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha}),$$

Finally, the solution of Helmholtz equation (4.1) is given by

$$H(x,y)=\lim_{m\to\infty}H_m(x,y)$$

$$=\lim_{m\to\infty}\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha})$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cosh_{\alpha}(y^{\alpha}). \tag{4.4}$$

Example 2. Consider the following Helmholtz equation involving local fractional operator in the form:

$$\frac{\partial^{2\alpha}H(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}H(x,y)}{\partial y^{2\alpha}}+H(x,y)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}, \tag{4.5}$$

subject to the initial value

$$H(0,y)=0,\ \frac{\partial^{\alpha}H(0,y)}{\partial x^{\alpha}}=\frac{y^{\alpha}}{\Gamma(1+\alpha)}. \tag{4.6}$$

In view of (3.11) and (4.5) the local fractional iteration algorithm can be written as follows:

$$H_{m+1}(x,y)=H_m(x,y)$$
$$-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}H_m(\xi,y)}{\partial\xi^{\alpha}}+\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{\frac{\partial^{2\alpha}H_m(\xi,y)}{\partial y^{2\alpha}}+H_m(\xi,y)\right\}\right)\right)-H_m^{(\alpha)}(0,y)-\frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}\right)(d\xi)^{\alpha} \tag{4.7}$$

We can use the initial conditions to select $H_0(x,y)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}$. Using this selection into the correction functional (4.7) gives the following successive approximations

$$H_0(x,y)=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)},$$

$$H_1(x,y)=H_0(x,y)$$
$$-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}H_0(\xi,y)}{\partial\xi^{\alpha}}+\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{\frac{\partial^{2\alpha}H_0(\xi,y)}{\partial y^{2\alpha}}+H_0(\xi,y)\right\}\right)\right)-H_0^{(\alpha)}(0,y)-\frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}\right)(d\xi)^{\alpha}$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{y^{\alpha}}{\Gamma(1+\alpha)}+\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}}Ł_{\alpha}\left\{0+\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}\right\}\right)\right)-\frac{y^{\alpha}}{\Gamma(1+\alpha)}-\frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}\right)(d\xi)^{\alpha}$$

$$=\frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^{\alpha}}{\partial\xi^{\alpha}}\left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{4\alpha}}\frac{y^{\alpha}}{\Gamma(1+\alpha)}\right)\right)-\frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}\right)(d\xi)^{\alpha}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(1+\alpha)}\int_0^x \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} \left(\frac{\xi^{3\alpha}}{\Gamma(1+3\alpha}\frac{y^{\alpha}}{\Gamma(1+\alpha)} \right) - \frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)} \right)(d\xi)^{\alpha}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)},$$

$$H_2(x,y) = H_1(x,y)$$
$$- \frac{1}{\Gamma(1+\alpha)}\int_0^x \left(\frac{\partial^{\alpha} H_1(\xi,y)}{\partial \xi^{\alpha}} + \frac{\partial^{\alpha}}{\partial \xi^{\alpha}} \left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}} Ł_{\alpha} \left\{ \frac{\partial^{2\alpha} H_1(\xi,y)}{\partial y^{2\alpha}} + H_1(\xi,y) \right\} \right) \right) - H_1^{(\alpha)}(0,y) - \frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)} \right)(d\xi)^{\alpha}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)},$$

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$$H_m(x,y) = H_{m-1}(x,y)$$
$$- \frac{1}{\Gamma(1+\alpha)}\int_0^x \left(\frac{\partial^{\alpha} H_{m-1}(\xi,y)}{\partial \xi^{\alpha}} + \frac{\partial^{\alpha}}{\partial \xi^{\alpha}} \left(Ł_{\alpha}^{-1}\left(\frac{1}{s^{2\alpha}} Ł_{\alpha} \left\{ \frac{\partial^{2\alpha} H_{m-1}(\xi,y)}{\partial y^{2\alpha}} + H_{m-1}(\xi,y) \right\} \right) \right) - H_{m-1}^{(\alpha)}(0,y) - \frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)} \right)(d\xi)^{\alpha}$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}.$$

Finally, the solution of Helmholtz equation (4.5) is given by

$$H(x,y) = \lim_{m\to\infty} H_m(x,y)$$
$$= \lim_{m\to\infty} \frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}$$
$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)}\frac{y^{\alpha}}{\Gamma(1+\alpha)}. \tag{4.8}$$

Example 3. Let us consider the following system of local fractional coupled Helmholtz equations with local fractional derivative:

$$\begin{aligned} &\frac{\partial^{2\alpha} H_1(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} H_2(x,y)}{\partial y^{2\alpha}} - H_1(x,y) = 0, \\ &\frac{\partial^{2\alpha} H_2(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} H_1(x,y)}{\partial y^{2\alpha}} - H_2(x,y) = 0, \end{aligned} \tag{4.9}$$

subject to the initial conditions

$$\begin{aligned} &H_1(0,y) = 0,\ \frac{\partial^{\alpha} H_1(0,y)}{\partial x^{\alpha}} = E_{\alpha}(y^{\alpha}), \\ &H_2(0,y) = 0,\ \frac{\partial^{\alpha} H_2(0,y)}{\partial x^{\alpha}} = -E_{\alpha}(y^{\alpha}). \end{aligned} \tag{4.10}$$

Applying local fractional Laplace transform on Eq. (4.9) and using the initial conditions, we have

$$\text{Ł}_\alpha\{H_1(x,y)\} = \frac{1}{s^{2\alpha}}E_\alpha(y^\alpha) + \frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_1(x,y) - \frac{\partial^{2\alpha}H_2(x,y)}{\partial y^{2\alpha}}\right\},$$
$$\text{Ł}_\alpha\{H_2(x,y)\} = -\frac{1}{s^{2\alpha}}E_\alpha(y^\alpha) + \frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_2(x,y) - \frac{\partial^{2\alpha}H_1(x,y)}{\partial y^{2\alpha}}\right\}. \tag{4.11}$$

Operating with the local fractional Laplace transform inverse on both sides of Eq. (4.11) we obtain

$$H_1(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha) + \text{Ł}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_1(x,y) - \frac{\partial^{2\alpha}H_2(x,y)}{\partial y^{2\alpha}}\right\}\right),$$
$$H_2(x,y) = -\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha) + \text{Ł}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_2(x,y) - \frac{\partial^{2\alpha}H_1(x,y)}{\partial y^{2\alpha}}\right\}\right). \tag{4.12}$$

Derivative both sides Eq. (4.12) with respect to x, we get

$$\frac{\partial^\alpha H_1(x,y)}{\partial x^\alpha} = E_\alpha(y^\alpha) + \frac{\partial^\alpha}{\partial x^\alpha}\text{Ł}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_1(x,y) - \frac{\partial^{2\alpha}H_2(x,y)}{\partial y^{2\alpha}}\right\}\right),$$
$$\frac{\partial^\alpha H_2(x,y)}{\partial x^\alpha} = -E_\alpha(y^\alpha) + \frac{\partial^\alpha}{\partial x^\alpha}\text{Ł}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_2(x,y) - \frac{\partial^{2\alpha}H_1(x,y)}{\partial y^{2\alpha}}\right\}\right). \tag{4.13}$$

Making the correction function is given

$$H_{1(m+1)}(x,y) = H_{1m}(x,y) - \frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_{1m}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(\text{Ł}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_{1m}(\xi,y) - \frac{\partial^{2\alpha}H_{2m}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) - E_\alpha(y^\alpha)\right)(d\xi)^\alpha,$$
$$H_{2(m+1)}(x,y) = H_{2m}(x,y) - \frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_{2m}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(\text{Ł}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_{2m}(\xi,y) - \frac{\partial^{2\alpha}H_{1m}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) + E_\alpha(x^\alpha)\right)(d\xi)^\alpha.$$

(4.14)

We can use the initial conditions to select $H_{10}(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha)$, $H_{20}(x,y) = -\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha)$. Using this selection into the correction functional (4.14) gives the following successive approximations:

$$H_{10}(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha),$$
$$H_{20}(x,y) = -\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(y^\alpha).$$

$$H_{11}(x,y) = H_{10}(x,y) - \frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_{10}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(\text{Ł}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_{10}(\xi,y) - \frac{\partial^{2\alpha}H_{20}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) - E_\alpha(y^\alpha)\right)(d\xi)^\alpha$$
$$H_{21}(x,y) = H_{20}(x,y) - \frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_{20}(\xi,y)}{\partial \xi^\alpha} - \frac{\partial^\alpha}{\partial \xi^\alpha}\left(\text{Ł}_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}\text{Ł}_\alpha\left\{H_{20}(\xi,y) - \frac{\partial^{2\alpha}H_{10}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right) + E_\alpha(x^\alpha)\right)(d\xi)^\alpha$$

$$= E_\alpha(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{2x^{3\alpha}}{\Gamma(1+3\alpha)}\right),$$

$$= -E_\alpha(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{2x^{3\alpha}}{\Gamma(1+3\alpha)}\right),$$

$$H_{12}(x,y)=H_{11}(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_{11}(\xi,y)}{\partial\xi^\alpha}-\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_{11}(\xi,y)-\frac{\partial^{2\alpha}H_{21}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)-E_\alpha(y^\alpha)\right)(d\xi)^\alpha$$

$$H_{22}(x,y)=H_{21}(x,y)-\frac{1}{\Gamma(1+\alpha)}\int_0^x\left(\frac{\partial^\alpha H_{21}(\xi,y)}{\partial\xi^\alpha}-\frac{\partial^\alpha}{\partial\xi^\alpha}\left(Ł_\alpha^{-1}\left(\frac{1}{s^{2\alpha}}Ł_\alpha\left\{H_{21}(\xi,y)-\frac{\partial^{2\alpha}H_{11}(\xi,y)}{\partial y^{2\alpha}}\right\}\right)\right)+E_\alpha(x^\alpha)\right)(d\xi)^\alpha$$

$$=E_\alpha(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{2x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{4x^{5\alpha}}{\Gamma(1+5\alpha)}\right),$$

$$=-E_\alpha(y^\alpha)\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{2x^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{4x^{5\alpha}}{\Gamma(1+5\alpha)}\right).$$

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$$H_{1m}(x,y)=E_\alpha(y^\alpha)\sum_{k=0}^{m}\frac{2^k x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)},$$

$$H_{2m}(x,t)=-E_\alpha(y^\alpha)\sum_{k=0}^{m}\frac{2^k x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}.$$

Therefore, the series solutions can be written in the form

$$\begin{aligned}H_1(x,y)&=\lim_{m\to\infty}H_{1m}(x,y)=E_\alpha(y^\alpha)\frac{\sinh_\alpha\left(\sqrt{2}x^\alpha\right)}{\sqrt{2}},\\ H_2(x,y)&=\lim_{m\to\infty}H_{2m}(x,y)=-E_\alpha(y^\alpha)\frac{\sinh_\alpha\left(\sqrt{2}x^\alpha\right)}{\sqrt{2}}.\end{aligned}\qquad(4.15)$$

Conclusions

In this work we considered the coupling method of the local fractional variational iteration method and Laplace transform to solve Helmholtz and coupled Helmholtz equations and their approximate solutions were obtained. The results include an efficient implement of the local fractional variational iteration transform method to solve the partial differential equations with local fractional derivative operator.

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