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Smarandache zero divisors, power joined elements and triple identity rings

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Abstract

In this paper, we study Smarandache zero divisors, in $\mathbb{Z}_n$ and the group ring $\mathbb{Z}_2 G$, for a cyclic group G of order 2n, we also study power joined elements and triple identity rings.

Introduction

Smarandache algebraic structures introduced by Raul Padilla and Florentine Smarandache [3] and [5]. Smarandache zero divisor, Smarandache nilpotent and Smarandache idempotent elements introduced by W. B. Vasantha Kandasamy [7]. This work consists of three sections. In section one we state basic definitions on Smarandache algebraic structure and some results that we need in our work. In section two we study Smarandache zero divisors in $\mathbb{Z}_n$ and in the group ring $\mathbb{Z}_2 G$, where G is a cyclic group of order $2n$. In section three we study Smarandache power joined elements and Smarandache power joined rings. It is shown that, the ring $\mathbb{Z}_n$ with the prime factorization of $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_m}^{\alpha_m} q$, is a Smarandache power joined ring. Furthermore, we study triple identity rings.

BACKGROUND

Definition 1.1 [7]. A ring $\mathcal{R}$ is said to be $Smarandache\ ring$ (S-ring) if $\mathcal{R}$ has a proper subset F, which is a field. If $\mathcal{S}$ is a subring of $\mathcal{R}$, then $\mathcal{S}$ is said to be a $Smarandache\ subring$ (S-subring) of $\mathcal{R}$ if itself is a Smarandache ring.

Definition 1.2 [6]. Let $\mathcal{R}$ be a ring and x, y be nonzero elements. Each of x and y in $\mathcal{R}$ is a $Smarandache\ zero\ divisor$ (S-zero divisor) if $xy = 0$ and there exist $a, b \in \mathcal{R} \setminus \{0, x, y\}$ with

1. $xa = 0 \text{ or } ax = 0.$
2. $yb = 0 \text{ or } by = 0.$
3. $ab \neq 0 \text{ or } ba \neq 0.$

Definition 1.3 [7]. Let $\mathcal{R}$ be a ring. An element $0 \neq x \in \mathcal{R}$ is said to be a $Smarandache\ idempotent$ (S-idempotent) of $\mathcal{R}$, if

1. $x^2 = x$
2. There exists $a \in \mathcal{R} \setminus \{x, 1, 0\}$, such that
   - i. $a^2 = x$ and
   - ii. $xa = a\ (ax = a)$or $ax = x\ (xa = x)$.

Definition 1.4 [7]. Two elements x and y of a ring $\mathcal{R}$ are said to be $power\ joined\ elements$ if there exist $m, n \in \mathbb{Z}^+$ such that $x^m = y^n$. If for every $0 \neq x \in \mathcal{R}$, there exists $y \in \mathcal{R}$, $(y \neq x)$ such that x and y are power joined elements, then $\mathcal{R}$ is said to be a $power\ joined\ ring$.

Definition 1.5 [7]. Let $\mathcal{R}$ be a ring in which for every $0 \neq x \in \mathcal{R}$, there exists $y \in \mathcal{R}$, such that $x^m = y^m$ $(x \neq y)$ and $m \geq 2$. Then $\mathcal{R}$ is called a $uniformly\ \ power\ joined\ ring$.

Definition 1.6 [7]. Let $\mathcal{R}$ be a ring. If for every $0 \neq a \in A \subset \mathcal{R}$, where A is Smarandache subring, there exists $b \in A$, such that $a^m = b^n$, for some positive integers m and n, then $\mathcal{R}$ is called a $Smarandache\ power\ joined\ ring$ (S-power joined ring). If $m = n$, $m \geq 2$, then $\mathcal{R}$ is called a $Smarandache\ uniformly\ power\ joined$ ring (S-uniformly power joined ring).

Definition 1.7 [7]. Let $\mathcal{R}$ be a ring. If there exists a triple $u, v, \omega \in \mathcal{R} \setminus \{0\}$ such that u, v and ω are distinct elements of $\mathcal{R} \setminus \{0\}$, which satisfy the identity $v^n + \omega^n = u^n$ $(n > 1)$, then $\mathcal{R}$ is called a $triple\ identity\ ring$ or TI-ring.

Definition 1.8 [7]. Let $\mathcal{R}$ be a ring, then $\mathcal{R}$ is a $Smarandache\ triple\ identity\ ring$ or (S-TI-ring) if $\mathcal{R}$ has a Smarandache subring A, and in A we have three distinct elements u, v, ω such that $v^n + \omega^n = u^n$.

Smarandache zero divisors

Proposition 2.1. In $\mathbb{Z}_{p^n}$, p is prime and $n \geq 3$, the Smarandache zero divisors are of the form p^2k for $1 \leq k \leq p^{n-2} - 1$.

Proof. Clearly p^2k is a zero divisor and $(p^2k)(p^{n-1}) \equiv 0 \pmod{p^n}$ for each k. Take $a = p^{n-2}$ and $b = p$.

$(p^2k)\, a = (p^2k)\, p^{n-2} \equiv 0 \pmod{p^n}$,

$p^{n-1}\, b = (p^{n-1})\, p \equiv 0 \pmod{p^n}$,

$ab = (p^{n-2})\ p = p^{n-1} \not\equiv 0 \pmod{p^n}$.

Thus, p^2k is a Smarandache zero divisor for $1 \leq k \leq p^{n-2} - 1$ in $\mathbb{Z}_{p^n}$, $(n \geq 3)$.

Note that if a zero divisor is not divisible by p^2, then it is not a Smarandache zero divisor.■

Proposition 2.2. [6] If $n = p_1p_2$ where p_1, p_2 are primes, then $\mathbb{Z}_n$ has no Smarandache zero divisor.

Theorem 2.3. If $n = pqr$, where p,q and r are distinct primes, then a Smarandache zero divisor x of $\mathbb{Z}_n$ has one of the forms $pq\ell$, $1 \leq \ell \leq r - 1$, prm, $1 \leq m \leq q - 1$ and qrs, $1 \leq s \leq p - 1$.

Proof. Clearly $pq\ell, prm$ and qrs are zero divisors for $1 \leq \ell \leq r - 1$, $1 \leq m \leq q - 1$ and $1 \leq s \leq p - 1$.

It is enough to show that $x = pq\ell$ is a Smarandache zero divisor, for $1 \leq \ell \leq r-1$.

For $y = prt$, $1 \leq t \leq q-1$, $xy = 0$.

Put $a = r$, $b = q$.

So, $xa = pq\ell\, r \equiv 0 \pmod{n}$,

$yb = prt\, q \equiv 0 \pmod{n}$,

and $ab = r\, q \not\equiv 0 \pmod{n}$.

So $pq\ell$ is a Smarandache zero divisor. Now, suppose $p \mid x$, $q \nmid x$ and $r \nmid x$. Then $x = tp$ for some $t \in \mathbb{Z}$ with $q \nmid t$ and $r \nmid t$. So if $yx = 0$, then $qr|y$, hence it is impossible to find a, b such that $ax = 0$ and $by = 0$ and $ab \neq 0$.■

Note that the number of Smarandache zero divisors of $\mathbb{Z}_n$, where $n = pqr$, p,q and r are distinct primes is $p + q + r - 3$.

Theorem 2.4. In $\mathbb{Z}_n$, with the prime factorization of $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, $\alpha_i \geq 0$, an element of the form $p_i\ell \pmod{n}$ for $1 \leq i \leq k$ is a Smarandache zero divisor if ℓ is divisible by one of the p_i's.

Proof. Clearly $p_i\ell$ is a zero divisor, for each i.

Now, suppose ℓ is divisible by p_j for some $1 \leq j \leq k$. We have to show that $p_i\ell$ is a Smarandache zero; there are two cases:

Case 1: $p_i = p_j$.

Put $x = p_i\,\ell$ which means $x = p_i^2 M$ ($M = \frac{\ell}{p_i}$).

Take $y = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k}$.

Clearly $xy \equiv 0 \pmod{n}$.

Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 2} p_{i+1}^{\alpha i+1} \ldots p_k^{\alpha_k}$ and $b = p_i$, then:

$ax = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 2} p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k} p_i^2 M \equiv 0 \pmod{n}$

and $by = p_i\, p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k} \equiv 0 \pmod{n}$

and $ab = p_1^{\alpha_1} \ldots p_i^{\alpha_i - 2} \ldots p_k^{\alpha_k} p_i = p_1^{\alpha_1} \ldots p_i^{\alpha_i - 1} \ldots p_k^{\alpha_k} \not\equiv 0 \pmod{n}$.

Hence x is a Smarandache zero divisor.

Case 2: $p_i \neq p_j$.

Put $x = p_i\,\ell$, where $\ell = p_j\, N$ for some positive integer N.

Take $y = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{j-1}^{\alpha_{j-1}} p_j^{\alpha_j - 1} p_{j+1}^{\alpha_{j+1}} \ldots p_k^{\alpha_k}$.

Clearly $xy \equiv 0 \pmod{n}$.

Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i - 1} \ldots p_j^{\alpha_j - 1} \ldots p_k^{\alpha_k}$ $i \neq j$ and $b = p_j$, then:

$ax = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \ldots {p_i}^{\alpha_i - 1} \ldots {p_j}^{\alpha_j - 1} \ldots {p_k}^{\alpha_k}\, p_i p_j\, N \equiv 0 \pmod{n}$,

$by = p_j\, {p_1}^{\alpha_1}{p_2}^{\alpha_2} \ldots {p_{j-1}}^{\alpha_{j-1}}{p_j}^{\alpha_j - 1}{p_{j+1}}^{\alpha_{j+1}} \ldots {p_k}^{\alpha_k} \equiv 0 \pmod{n}$

and $ab = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \ldots {p_i}^{\alpha_i - 1} \ldots {p_j}^{\alpha_j} \ldots {p_k}^{\alpha_k} \not\equiv 0 \pmod{n}$.

Hence $p_i\,\ell$ is a Smarandache zero divisor if ℓ is divisible by one of the p_i's.■

As in **Theorem 2.3**, if ℓ is not divisible by p_j for each $1 \le j \le k$, then $p_i\,\ell$ is not a Smarandache zero divisor.

Proposition 2.5. Let $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$ be integral domains with $n > 2$. A nonzero element $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\mathcal{R}_1 \times \mathcal{R}_2 \times \ldots \times \mathcal{R}_n$ is a Smarandache zero divisor if there exist i, j with $i \neq j$ such that $\alpha_i = \alpha_j = 0$.

Proof. Without loss of generality we can assume $\alpha_1 = \alpha_2 = 0$, not all $\alpha_i = 0$, this means that $x = (0, 0, \alpha_3, \ldots, \alpha_n)$. Let $y = (0, \beta_1, 0, \ldots, 0)$, $\beta_1 \neq 0$, that is $xy = 0$. Take $a = (\gamma_1, 0, \ldots, 0)$ and $b = (\eta_1, 0, \ldots, 0)$ with $\gamma_1, \eta_1, \neq 0$. So $xa = (0, \ldots, 0)$ and $yb = (0, \ldots, 0)$. But $ab = (\gamma_1\eta_1, 0, \ldots, 0)$.

Hence $(0, 0, \alpha_3, \ldots, \alpha_n)$ is a Smarandache zero divisor.■

Now, we discuss Smarandache zero divisors in $\mathbb{Z}_2 G$, where G is a cyclic group. The following lemma is needed.

Lemma 2.6. Consider the group ring $\mathbb{Z}_2G$, where G is a cyclic group of order m. Then, for each $1 \le \ell \le m$, $(1 + g^{\ell})(1 + g + g^2 + \ldots + g^{m-1}) = 0$.

Theorem 2.7. Let $\mathbb{Z}_2 G$ be a group ring, where $G = \langle g : g^{2n} = 1 \rangle$ is a cyclic group of order $2n$. If $2 \le \ell \le 2n - 1$ with $\gcd(\ell, 2n) = d \neq 1$ then $1 + g^{\ell}$ is a Smarandache zero divisor. Moreover, $g^k(1 + g^{\ell})$ is a Smarandache zero divisor, for $1 \le k \le 2n - \ell - 1$.

Proof. Since $\gcd(\ell, 2n) = d \neq 1$. So, $\ell = kd$ and $2n = sd$ for some $k, s \in \mathbb{N}$. By **Lemma 2.6**, $(1 + g^{\ell})$ is a zero divisor and $xy = 0$ where $x = 1 + g^{\ell}$ and $y = 1 + g + g^2 + \cdots + g^{2n-1}$.

Take, $a = g + g^{1+d} + g^{1+2d} + g^{1+3d} + \cdots + g^{1+2n-d}$ and $b = 1 + g$.

Let us find xa in $\mathbb{Z}_2G$.

$$xa = (1 + g^{kd})(g + g^{1+d} + g^{1+2d} + \cdots + g^{1+(k-1)d} + g^{1+kd} + g^{1+(k+1)d} + \cdots + g^{1+(s-k)d} + g^{1+(s-k+1)d} + \cdots + g^{1+(s-1)d}).$$

$$= g + g^{1+d} + g^{1+2d} + \cdots + g^{1+(k-1)d} + g^{1+kd} + g^{1+(k+1)d} + \cdots + g^{1+(s-k-1)d} + g^{1+(s-k)d} + g^{1+(s-k+1)d} + \cdots + g^{1+(s-1)d} + g^{1+kd} + g^{1+(k+1)d} + \cdots + g^{1+(s-1)d} + g^{1+sd}\, g + g^{1+(s+1)d} + \cdots + g^{1+(s+k-1)d} = 0,$$

$$yb = (1 + g + g^2 + \ldots + g^{\ell-1} + g^{\ell} + \ldots + g^{2n-1})(1 + g)$$

$$= 1 + g + g^2 + \ldots + g^{\ell-1} + g^{\ell} + \ldots + g^{2n-1} + g + g^2 + \ldots + g^{\ell-1} + g^{\ell} + g^{\ell+1} + \ldots + g^{2n-1} + 1 = 0,$$

and

$ab \neq 0$.

Hence, $(1+\mathrm{g}^{\ell})$ is a Smarandache zero divisor .■

Note that for each ℓ the number of Smarandache zero divisors, which are of the form $\mathrm{g}^{k}\,(1+\mathrm{g}^{\ell})$, $0 \leq k \leq 2n-\ell-1$ is $2n-\ell$ and the total number of Smarandache zero divisors of this form is

$$\sum_{\substack{\ell=2 \\ \gcd(\ell,2n)\neq 1}}^{2n-1} (2n-\ell).$$

Example 2.1. Let $n=4$. The Smarandache zero divisors of the form $\mathrm{g}^{k}\,(1+\mathrm{g}^{\ell})$ for $0 \leq k \leq 2n-\ell-1$ and gcd(ℓ, $2n$)$\neq 0$ are the following 12 elements

$1+\mathrm{g}^2$, $\mathrm{g}(1+\mathrm{g}^2)$, $\mathrm{g}^2(1+\mathrm{g}^2)$, $\mathrm{g}^3(1+\mathrm{g}^2)$, $\mathrm{g}^4(1+\mathrm{g}^2)$, $\mathrm{g}^5(1+\mathrm{g}^2)$

$1+\mathrm{g}^4$, $\mathrm{g}(1+\mathrm{g}^4)$, $\mathrm{g}^2(1+\mathrm{g}^4)$, $\mathrm{g}^3(1+\mathrm{g}^4)$

$1+\mathrm{g}^6$, $\mathrm{g}(1+\mathrm{g}^6)$.

Remark 1. In the group ring $\mathbb{Z}_2G$ under the conditions given in **Theorem 2.7**, $x=\mathrm{g}+\mathrm{g}^{1+d}+\mathrm{g}^{1+2d}+\cdots+\mathrm{g}^{1+2n-d}$ is a Smarandache zero divisor.

Put $y=1+\mathrm{g}+\mathrm{g}^2+\ldots+\mathrm{g}^{\ell-1}+\mathrm{g}^{\ell}+\ldots+\mathrm{g}^{2n-1}$.

Clearly $xy=0$.

Take $a=1+\mathrm{g}^{\ell}$ and $b=1+\mathrm{g}$.

So, $ax=0$ and $by=0$, but $ab=(1+\mathrm{g}^{\ell})\ (1+\mathrm{g})= 1+\mathrm{g}^{\ell}+\mathrm{g}+\mathrm{g}^{\ell}+1 \not\equiv 0$. Hence, x is a Smarandache zero divisor.

Proposition 2.8. Every Smarandache nilpotent element in a ring $\mathcal{R}$ is a Smarandache zero divisor.

Proof. Let x be a Smarandache nilpotent element. So $x^n=0$, for some $n>1$, suppose n is the least positive integer such that $x^n=0$. Then x is a zero divisor and there exists $y \in \mathcal{R}\backslash\{0,x\}$ such that $y^k \neq 0$, for each k and $yx^{\ell}=0$, for $\ell<n$. Now we have $x.x^{n-1}=0$. Take $a=y\,x^{\ell-1}$ and $b=y$. Hence

$b\,x^{n-1}=b\,x^{\ell}\,x^{n-1-\ell}=0$ and

$a\,x=\ y\,x^{\ell-1}\,x=0$ and

$ab=y\,x^{\ell-1}\,y=y^2\,x^{\ell-1}\neq 0$. Therefore, x is a Smarandache zero divisor.■

Remark 2. Clearly an idempotent element in a ring $\mathcal{R}$ is a zero divisor, but a Smarandache idempotent element need not be a Smarandache zero divisor, in general. For example, in the ring $\mathbb{Z}_{15}$, as shown in [2] the only nontrivial idempoents are 6 and 10, which are zero divisors, but none of them is a Smarandache zero divisor.■

Smarandache power joined elements and triple identity rings

Proposition 3.1. $\mathbb{Z}_{p^n}$ is a power joined ring, where p is a prime and $n \geq 2$.

Proof. Let $x \in \mathbb{Z}_{p^n}$. If $p \nmid x$ then by **Euler Theorem** [1], $x^{\varphi(p^n)} \equiv 1 \pmod{p^n}$ and if $p \mid x$, so $x = kp$ for $1 \le k \le p^{n-1} - 1$, then $x^n \equiv 0 \pmod{p^n}$. Hence $\mathbb{Z}_{p^n}$ is a power joined ring.■

Proposition 3.2. The ring $\mathbb{Z}_p$, where p is an odd prime, is a uniformly power joined ring.

Proof. Let $x, y \in \mathbb{Z}_p$ such that $x \neq y$, then by **Euler Theorem** $x^{\varphi(p)} \equiv y^{\varphi(p)} \equiv 1 \pmod{p}$, which means that $\mathbb{Z}_p$ is a uniformly power joined ring.■

Theorem 3.3. Consider $\mathbb{Z}_m$, with $m = p^\alpha q$, (p, q are distinct primes), $\alpha \ge 2$. Then $\mathbb{Z}_m$ is a power joined ring.

Proof. Let $x \in \mathbb{Z}_m$. There are three cases.

***Case 1*:** $pq \mid x$, that is x is nilpotent, then $x^\alpha \equiv 0^\alpha \pmod{m}$.

Case 2: i) $0 < x < pq$ and $q \mid x$, so $x = rq$. Put $y = x + pq$. Then

$$y^{\varphi(p^\alpha)} \equiv \left(q(r+p)\right)^{\varphi(p^\alpha)}$$

$$\equiv q^{\varphi(p^\alpha)}(r^{\varphi(p^\alpha)} + \varphi(p^\alpha)\, r^{\varphi(p^\alpha)-1}\, p + \frac{\varphi(p^\alpha)\,(\varphi(p^\alpha)-1)}{2!}\, r^{\varphi(p^\alpha)-2}\, p^2$$

$$+ \cdots + \varphi(p^\alpha)\, r\, p^{\varphi(p^\alpha)-1} + p^{\varphi(p^\alpha)}) \pmod{m}$$

$$\equiv q^{\varphi(p^\alpha)} r^{\varphi(p^\alpha)} \pmod{m} \equiv x^{\varphi(p^\alpha)} \pmod{m}.$$

ii) $0 < x < pq$ and $p \mid x$, so $x = sp$. Put $y = x + pq$. Then

$$y^{\varphi(q)} \equiv \left(p(s+q)\right)^{\varphi(q)}$$

$$\equiv p^{\varphi(q)}(s^{\varphi(q)} + \varphi(q)\, s^{\varphi(q)-1}\, q + \frac{\varphi(q)\,(\varphi(q)-1)}{2!}\, s^{\varphi(q)-2}\, q^2 + \cdots + \varphi(q)\, s\, q^{\varphi(q)-1} + p^{\varphi(q)}) \pmod{m}$$

$$\equiv p^{\varphi(q)} s^{\varphi(q)} \pmod{m} \equiv x^{\varphi(q)} \pmod{m}.$$

Similarly, for other cases which are $(t-1)pq \le x \le tpq$, $1 \le t \le p^{\alpha-1}$.

***Case* 3)** If $p \nmid x$ and $q \nmid x$. In this case take $0 \neq y \in \mathbb{Z}_m$ such that gcd $(y, pq) = 1$, hence $x^{\varphi(m)-1} \equiv y^{\varphi(m)-1} \equiv 1 \pmod{m}$.

So $\mathbb{Z}_m$ is a power joined ring.■

Theorem 3.4. Consider $\mathbb{Z}_m$, with $m = p^\alpha q$, (p, q are distinct primes), $\alpha \ge 2$. Then $\mathbb{Z}_m$ is a Smarandache power joined ring.

Proof. Let $\mathcal{A} = \{0, p, 2p, \dots, p(p^{\alpha-1}q - 1)\} \subset \mathbb{Z}_m$. Clearly $\mathcal{A}$ is a subring of $\mathbb{Z}_m$. It remains to show that $\mathcal{A}$ contains a proper subset, which is a field. Let $\mathcal{H}$ be the principal ideal of $\mathbb{Z}_m$ generated by p^α, that is $\mathcal{H} = \{0, p^\alpha, 2p^\alpha, \dots, (q-1)p^\alpha\}$, which is a field [4]. Hence $\mathcal{A}$ is a Smarandache subring. Now it remains to show that, for each $x \in \mathcal{A}$, there is $y \in \mathcal{A}$ such that $x^\alpha = y^\alpha \pmod{m}$.

Let $x \in \mathcal{A}$. Then $x = r\,p$, for some $0 \le r \le p^{\alpha-1}q - 1$. So there are three cases for r, either $r = q$ or $r < q$ or $r > q$.

Case 1: $r = q$ this means that $x = pq$, take $y = 0$, then

$$x^{\alpha} = y^{\alpha} \text{ (mod } m).$$

Case 2: $r < q$ or $q < r$. Suppose $r < q$. Put $= pq + pr$. Then

$y^{\alpha} = \left(p(q + r)\right)^{\alpha}$

$= p^{\alpha}(q^{\alpha} + \alpha\, q^{\alpha-1}\, r + \frac{\alpha\,(\alpha-1)}{2!}\, q^{\alpha-2}\, r^{2} + \cdots + \alpha\, q\, r^{\alpha-1} + r^{\alpha})$ (mod m)

$= p^{\alpha} r^{\alpha} = x^{\alpha}$, then

$$x^{\alpha} = y^{\alpha} \text{ (mod } m).$$

Similarly, for $q < r$. Therefore, $\mathbb{Z}_m$ is a Smarandache power joined ring.■

Theorem 3.5. Consider $\mathbb{Z}_n$,with $n = {p_1}^{\alpha_1}\, {p_2}^{\alpha_2} \ldots {p_m}^{\alpha_m}\, q$, (p_i, q are distinct primes), $\alpha_i \geq 2$ $(1 \leq i \leq m)$, then $\mathbb{Z}_n$ is a Smarandache power joined ring.

Proof. Let $\mathcal{A} = \{0, p_1\, p_2 \ldots p_m, 2p_1\, p_2 \ldots p_m, \ldots, n - p_1\, p_2 \ldots p_m\} \subset \mathbb{Z}_n$, and clearly $\mathcal{A}$ is a subring of $\mathbb{Z}_n$. Let $\mathcal{H}$ be the principal ideal of $\mathbb{Z}_n$ generated by $\gamma = {p_1}^{\alpha_1}\, {p_2}^{\alpha_2} \ldots {p_m}^{\alpha_m}$,that is $\mathcal{H}$ ={0, γ, 2 γ,…, (q−1) γ}. It is shown in [4] that $\mathcal{H}$ is a field. Therefore $\mathcal{A}$ is a Smarandache subring. Now, it remains to show that for each $x \in \mathcal{A}$ there is $y \in \mathcal{A}$ such that $x^{\alpha} \equiv y^{\alpha}$ (mod n).

Let $x \in \mathcal{A}$. Then $x = r\, p_1\, p_2 \ldots p_m$ for some r. There are three cases for r, either $r = q$ or $r < q$ or $r > q$.

Case 1: $r = q$ this means that $x = p_1\, p_2 \ldots p_m q$, take $y = 0$ then

Then $x^{\alpha} = y^{\alpha}$ (mod n) where $\alpha = \max\limits_{1 \leq i \leq m} \{\alpha_i\}$.

Case 2: $r < q$ or $q < r$. We prove for $r < q$.

Put $y = p_1\, p_2 \ldots p_m q + p_1\, p_2 \ldots p_m r$. Then

$y^{\alpha} = \left(p_1\, p_2 \ldots p_m(q + r)\right)^{\alpha}$ where $\alpha = \max\limits_{1 \leq i \leq m} \{\alpha_i\}$

$= (p_1\, p_2 \ldots p_m)^{\alpha}(q^{\alpha} + \alpha\, q^{\alpha-1}\, r + \frac{\alpha\,(\alpha-1)}{2!}\, q^{\alpha-2}\, r^{2} + \cdots + \alpha\, q\, r^{\alpha-1} + r^{\alpha})$ (mod n)

$= (p_1\, p_2 \ldots p_m)^{\alpha} r^{\alpha} = x^{\alpha}$, then

$$x^{\alpha} = y^{\alpha} \text{ (mod } n).$$

Therefore $\mathbb{Z}_n$ is a Smarandache power joined ring.■

In what remains we study triple identity rings.

Proposition 3.6. The ring $\mathbb{Z}_p$, where $p \geq 5$ prime, is a triple identity ring.

Proof. Let x , y be two nonzero elements of $\mathbb{Z}_p$ such that $x + y \neq 0$. We show that x , y and $x + y$ satisfy the identity $x^p + y^p = (x + y)^p$.

Now, $(x + y)^p$

$= x^p + p\,x^{p-1}\,y + \frac{p(p-1)}{2!}\,x^{p-2}\,y^2 + \cdots + p\,x\,y^{p-1} + y^p \pmod p$

$\equiv x^p + y^p \pmod p \equiv x + y$

and $x^p + y^p \equiv x + y \pmod p$ by **Euler Theorem.** Thus $\mathbb{Z}_p$ is a triple identity ring.■

Proposition 3.7. The ring $\mathbb{Z}_{p^2}$, where p prime, is a triple identity ring.

Proof. Clearly p, $p+1$, $p^2-(p+1)$ are distinct elements in $\mathbb{Z}_{p^2}\setminus\{0\}$. We show that this triple satisfy the identity

$p^2 + (p+1)^2 = (p^2-(p+1))^2$.

L.H.S = $p^2 + (p+1)^2 = p^2 + p^2 + 2p + 1 \equiv 2p + 1 \pmod{p^2}$.

R.H.S = $(p^2-(p+1))^2$

$= p^4 - 2\,p^2(p+1) + (p+1)^2 \equiv 2p + 1 \pmod{p^2}$

Hence, $\mathbb{Z}_{p^2}$ is a triple identity ring.■

In general we have

Proposition 3.8. The ring $\mathbb{Z}_{p^n}$, p prime and $n \geq 3$, is a triple identity ring.

Proof. We will show that the triple p, $2p$ and $3p$ of distinct elements in $\mathbb{Z}_{p^n}\setminus\{0\}$ satisfy the identity

$p^n + (2p)^n = (3p)^n$.

Now, $p^n + (2p)^n \equiv (1+2^n)p^n \equiv 0 \pmod{p^n}$.

and $(3p)^n \equiv 0 \pmod{p^n}$.

Which means that $\mathbb{Z}_{p^n}$ is a triple identity ring.■

Lemma 3.9. If the group ring $\mathbb{Z}_2 G$, where G is an abelian group has two nontrivial idempotent elements, then $\mathbb{Z}_2 G$ is a triple identity ring.

Theorem 3.10. The group ring $\mathbb{Z}_2 G$ is a triple identity ring for a cyclic group G.

Proof: It is shown in [8] that the group ring $\mathbb{Z}_2 G$ contains at least two nontrivial idempotents. Now,

(1) If G is a cyclic group of an odd order $m > 1$, then

$\omega = g + g^2 + g^3 + \ldots + g^{\frac{m-1}{2}} + g^{\frac{m-1}{2}+1} + \ldots + g^{m-1}$ and $1+\omega$

are nontrivial idempotents . By ***Lemma*** **3.9**, $\mathbb{Z}_2 G$ is a triple identity ring.

(2) If G is a cyclic group of an even order $2n$, then either n is odd or n is even. If n is odd then $\omega = g^2 + g^4 + \ldots + g^{2n-2}$ and $1+\omega$ are nontrivial idempotents, so by ***Lemma*** **3.9**, $\mathbb{Z}_2 G$ is a triple identity ring.

If n is even, we consider the triple 1, g, $1+g$ in $\mathbb{Z}_2 G\setminus\{0\}$ to show that this triple satisfy the identity $1^{2n} + g^{2n} = (1+g)^{2n}$.

Now, $1^{2n} + g^{2n} = 1 + 1 = 0$.

and $(1 + g)^{2n} = 1 + g^{2n} = 0$.

Which means that $\mathbb{Z}_2 G$ is a triple identity ring.■

Theorem 3.11. Consider $\mathbb{Z}_n$, with $n = p^{\alpha} q$, (p, q are distinct primes), $\alpha \geq 2$. Then $\mathbb{Z}_n$ is a Smarandache triple ring.

Proof: As shown in **Theorem 3.4**, $\mathcal{A} = \{0, p, 2p, \dots, n - p\} \subset \mathbb{Z}_n$ is a Smarandache subring of $\mathbb{Z}_n$. Now, consider the distinct elements p, pq and $pq + p$ in $\mathcal{A}$. We will show that

$p^{\alpha} + pq^{\alpha} = (pq + p)^{\alpha}$.

Clearly $p^{\alpha} + pq^{\alpha} \equiv p^{\alpha} \pmod{n}$, and

$(pq + p)^{\alpha} = (pq)^{\alpha} + \alpha\,(pq)^{\alpha-1}p + \frac{\alpha(\alpha-1)}{2!}(pq)^{\alpha-2}p^2 + \cdots + \alpha(pq)(p)^{\alpha-1} + p^{\alpha} \equiv p^{\alpha} \pmod{n}$.

Hence $p^{\alpha} + pq^{\alpha} = (pq + p)^{\alpha}$, that is $\mathbb{Z}_n$ is a Smarandache triple ring.■

Finally, the following result is given.

Theorem 3.12. Consider $\mathbb{Z}_n$, with $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_m}^{\alpha_m} q$ (p_i, q are distinct primes), $\alpha_i \geq 2$ $(1 \leq i \leq m)$. Then $\mathbb{Z}_n$ is a Smarandache triple ring.

Proof: Let $\mathcal{A} = \{0, p_1 p_2 \dots p_m, 2p_1 p_2 \dots p_m, \dots, n - p_1 p_2 \dots p_m\} \subset \mathbb{Z}_n$. clearly $\mathcal{A}$ is a subring of $\mathbb{Z}_n$. Let $\mathcal{H}$ be the principal ideal of $\mathbb{Z}_n$ generated by $\gamma = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_m}^{\alpha_m}$,that is $\mathcal{H} = \{0, \gamma, 2\gamma, \dots, (q-1)\gamma\}$. It is shown in [4] that $\mathcal{H}$ is a field. Therefore $\mathcal{A}$ is a Smarandache subring. Consider the triple $p_1 p_2 \dots p_m$, $p_1 p_2 \dots p_m q$ and $p_1 p_2 \dots p_m q + p_1 p_2 \dots p_m$ which are distinct elements in $\mathcal{A}$ and let $\beta = \max_{1 \leq i \leq k}\{\alpha_i\}$.

Then

$(p_1 p_2 \dots p_m)^{\beta} + (p_1 p_2 \dots p_m q)^{\beta} \equiv (p_1 p_2 \dots p_m)^{\beta} \pmod{n}$,

and clearly $(p_1 p_2 \dots p_m q + p_1 p_2 \dots p_m)^{\beta} =$

$(p_1 p_2 \dots p_m q)^{\beta} + \beta\,(p_1 p_2 \dots p_m q)^{\beta-1}(p_1 p_2 \dots p_m) + \frac{\beta(\beta-1)}{2!}(p_1 p_2 \dots p_m q)^{\beta-2}(p_1 p_2 \dots p_m)^2 + \cdots +$
$\beta(p_1 p_2 \dots p_m q)(p_1 p_2 \dots p_m)^{\beta-1} + (p_1 p_2 \dots p_m)^{\beta} \equiv (p_1 p_2 \dots p_m)^{\beta} \pmod{n}$

Hence $(p_1 p_2 \dots p_m)^{\beta} + (p_1 p_2 \dots p_m q)^{\beta} = (p_1 p_2 \dots p_m q + p_1 p_2 \dots p_m)^{\beta} \pmod{n}$, that is $\mathbb{Z}_n$ is a Smarandache triple identity ring.■

REFERENCES

[1] D. M. Burton, **Elementary Number Theory,** Allyn and Bacon, Inc. (1980).

[2] P. A. Hummadi, **S-units and S-idempotents**, Zanco, Journal of Pure and Applied Sciences, Salahaddin University−Hawler, Vol. 21, No. 4, pp. 137-144. (2009).

[3] R. Padilla, **Smarandache Algebraic Structures**, Bulletin of Pure and Applied Sciences, Delhi, Vol. 17E., No.1, pp. 119-121. (1998).

[4] P. A. Hummadi and S. A. Osman, **Smarandache Rings and Smarandache Elements,** Raf. J. of Comp. & Math's, Vol. 10, No. 4, pp. 61-69. (2013).

[5] F. Smarandache, **Special Algebraic Structures**, in Collected Papers, Abaddaba, Oradea, Vol.3, pp. 78-81. (2000).

[6] W. B.Vasantha Kandasamy, **Smarandache Zero Divisors**. (2002).

http://www.gallup.unm.edu/~smarandache/ZeroDivisor.pdf

[7] W. B. Vasantha Kandasamy**, Smarandache Rings**, American Research Press. (2002).

[8] P. A. Hummadi and S. A. Osman, **Smarandache idempotents in certain types of group rings ,** Journal of Zankoy Sulaimani, 13(1) Part A , pp. 93-102. (2010).