



# The New Application of the Cauchy Operator

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## Abstract

In this paper, introduced new application of the Cauchy operator, where we define a  $q$ -polynomials  $Z_n(x; q)$ , then we represent this  $q$ -polynomials by Cauchy operator, to derive the identities of  $Z_n(x; q)$  in simple way, where we using this representation and the limit technique to introduce the basic identities: generating function, Mehler's formula and Roger's formula for  $Z_n(x; q)$  polynomials. Also we introduce an extension of the generating function, extension of Mehler's formula and extension of the Roger's formula. All identities will be derived in this paper depending on the roles of the Cauchy operator given in [1,11].

## Introduction and Notation

In 2008 , Chen and Gu [11] introduced the Cauchy augmentation operator for basic hypergeometric series, it was involves two parameters, so it can be consider as a general form of the  $q$ -exponential operator  $T(bD_q)$ . They found that the Cauchy operator is suitable for the study of some terminating summation formulas of basic hypergeometric series and  $q$ -series identities and  $q$ -integrals. The method of represent the  $q$ -series by operators was developed by Chen and Liu [7] and applied in more papers such as [1, 2, 3, 4, 6, 7, 9, 10, 12, 13, 14, 15, 16].

In this paper we will use the limit technique of the Cauchy operator to represent the special  $q$ -polynomials  $Z_n(x; q)$  and derive their generating function, Mehler's formula, Roger's formula, extended generating function, extended Mehler's formula and extended Roger's formula.

Let us review some common notation and terminology for basic hypergeometric series in [12]. Throughout this paper, we assume that  $|q| < 1$ , and  $q \neq 0$ , the  $q$ -shifted factorial is defined by:

$$(a; q)_0 = 1, \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{Z}^+.$$

where:

$$(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}. \tag{1.1}$$

The following notation refers to the multiple  $q$ -shifted factorials [12]:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \dots (a_m; q)_n,$$

Also:

$$(a_1, a_2, \dots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \dots (a_m; q)_\infty .$$

The  $q$  –binomial coefficients, or the Gaussian polynomials [12], are given by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} .$$

The basic hypergeometric series  ${}_{r+1}\phi_r$  are defined by [12]:

$${}_{r+1}\phi_r \left( \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, b_2, \dots, b_r; q)_n} x^n .$$

The Cauchy identity [12] is defined as:

$$\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} x^k = \frac{(ax; q)_\infty}{(x; q)_\infty}, \quad |x| < 1. \tag{1.2}$$

when  $a = 0$ , (1.2) gives the Euler’s identity [12]:

$$\sum_{k=0}^{\infty} \frac{x^k}{(q; q)_k} = \frac{1}{(x; q)_\infty}, \quad |x| < 1. \tag{1.3}$$

which has the following inverse relation [12]:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(q; q)_k} = (x; q)_\infty . \tag{1.4}$$

The  $q$  –difference operator  $D_q$  and the  $q$  –exponential operator  $T(bD_q)$  are defined in [8] as:

$$D_q\{f(a)\} = \frac{f(a) - f(aq)}{a}, \quad T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q; q)_n} .$$

**Proposition (1.1)**, [8]: Let  $D_q$  defined as above, then:

$$D_q^k\{a^n\} = \frac{(q; q)_n}{(q; q)_{n-k}} a^{n-k} . \tag{1.5}$$

Chen and Gu [11] defined the Cauchy operator as follows:

**Definition(1.2)**, [11]: Let  $D_q$  defined as above, then:

$$T(a, b; D_q) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (bD_q)^n .$$

Cauchy operator is the reminiscent of the Cauchy identity (1.2), they obtained the following operator identities where they assumed that the operator acts on the parameter  $c$ :

**Proposition (1.2)**, [11]: Let  $T(a, b; D_q)$  defined as above, then:

$$T(a, b; D_q)\{c^n\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (a; q)_k b^k c^{n-k} . \tag{1.6}$$

$$T(a, b; D_q) \left\{ \frac{1}{(ct; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, ct; q)_\infty}, \tag{1.7}$$

where  $|bt| < 1$ .

$$T(a, b; D_q) \left\{ \frac{1}{(cs, ct; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, cs, ct; q)_\infty} {}_2\phi_1 \left( \begin{matrix} a, ct \\ abt \end{matrix}; q, bs \right), \tag{1.8}$$

where  $\max\{|bs|, |bt|\} < 1$ .

$$T(a, b; D_q) \left\{ \frac{(cv; q)_\infty}{(cs, ct; q)_\infty} \right\} = \frac{(abs, cv; q)_\infty}{(bs, cs, ct; q)_\infty} {}_3\phi_2 \left( \begin{matrix} a, cs, v/t \\ abs, cv \end{matrix}; q, bt \right), \tag{1.9}$$

where  $\max\{|bs|, |bt|\} < 1$ .

In 2009, the author [1] used the Cauchy operator to derive the basic and extended identities for the homogeneous Rogers-Szegő polynomials, so he introduced the following operator identities, where the operator acts on a parameter  $c$ :

**Proposition(1.3)**, [1]: Let  $T(a, b; D_q)$  defined as above, then:

$$T(a, b; D_q) \left\{ \frac{c^n}{(ct; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, ct; q)_\infty} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, ct; q)_j}{(abt; q)_j} b^j c^{n-j}, \tag{1.10}$$

where  $|bt| < 1$ .

$$T(a, b; D_q) \left\{ \frac{c^n}{(cs, ct; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, cs, ct; q)_\infty} \sum_{l=0}^{\infty} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, ct; q)_{j+l} (cs; q)_j}{(abt; q)_{j+l} (q; q)_l} b^{j+l} c^{n-j} s^l, \tag{1.11}$$

where  $\max\{|bs|, |bt|\} < 1$ .

$$T(a, b; D_q) \left\{ \frac{c^n (cv; q)_\infty}{(cs, ct; q)_\infty} \right\} = \frac{(abs, cv; q)_\infty}{(bs, cs, ct; q)_\infty} \sum_{l=0}^{\infty} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{(a, cs; q)_{j+l} (ct; q)_j (v/t; q)_l}{(abs, cv; q)_{j+l} (q; q)_l} \times b^{j+l} c^{n-j} t^l, \tag{1.12}$$

where  $\max\{|bs|, |bt|\} < 1$ .

$$T(a, b; D_q) \left\{ \frac{1}{(cs, ct, cv; q)_\infty} \right\} = \frac{(abt; q)_\infty}{(bt, cs, ct, cv; q)_\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a, ct; q)_{k+j} (cs; q)_k (bs)^j (bv)^k}{(abt; q)_{k+j} (q; q)_j (q; q)_k}, \tag{1.13}$$

where  $\max\{|bs|, |bt|\} < 1$ .

$$T(a, b; D_q) \left\{ \frac{(cv, cw; q)_\infty}{(cs, ct, cu, cz; q)_\infty} \right\} = \frac{(abs, cv, cw; q)_\infty}{(bs, cs, ct, cu, cz; q)_\infty} \sum_{j,k,l=0}^{\infty} \frac{(a, cs; q)_{j+k+l} (ct; q)_{k+l}}{(abs, cv; q)_{j+k+l}} \times \frac{(cu, w/z; q)_k (v/t; q)_j (bt)^j (bz)^k (bu)^l}{(cw; q)_k (q; q)_j (q; q)_k (q; q)_l} \tag{1.14}$$

where  $\max\{|bs|, |bt|\} < 1$ .

### Operator Role and Basic Identities of $q$ –Polynomials

In this section, firstly we introduce new role of the Cauchy operator, then we state the definition of the  $q$  –polynomials  $Z_n(x; q)$  and represent this polynomials by a special case of the Cauchy operator. So we deriving the generating function, Mehler's formula and the Rogers formula of  $Z_n(x; q)$  depending on our representation and the roles of Cauchy operator.

**Theorem (2.1):** Let the Cauchy operator  $T(a, b; D_q)$  defined as above, we have:

$$T(a, b; D_q) \left\{ \frac{1}{(cs, ct, cu, cz, cv; q)_\infty} \right\} = \frac{(abs; q)_\infty}{(bs, cs, ct, cu, cz, cv; q)_\infty} \\ \times \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a, cs; q)_{k+i+j+l} (ct; q)_{k+j+l} (cu; q)_{k+j} (cz; q)_k}{(abs; q)_{k+i+j+l}} \frac{(bv)^k (bt)^i (bz)^j (bu)^l}{(q; q)_k (q; q)_i (q; q)_j (q; q)_l}, \tag{2.1}$$

where  $\max \{ |bs|, |bt| \} < 1$ .

*Proof.*

$$T(a, b; D_q) \left\{ \frac{1}{(cs, ct, cu, cz, cv; q)_\infty} \right\} = \sum_{n=0}^{\infty} \frac{(a; q)_n b^n}{(q; q)_n} D_q^n \left\{ \frac{1}{(cs, ct, cu, cz, cv; q)_\infty} \right\} \\ = \sum_{n=0}^{\infty} \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} D_q^k \left\{ \frac{1}{(cv; q)_\infty} \right\} D_q^{n-k} \left\{ \frac{1}{(csq^k, ctq^k, cuq^k, czq^k; q)_\infty} \right\} \\ = \sum_{n=0}^{\infty} \frac{(a; q)_n b^n}{(q; q)_n} \sum_{k=0}^n q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \frac{v^k}{(cv; q)_\infty} D_q^{n-k} \left\{ \frac{1}{(csq^k, ctq^k, cuq^k, czq^k; q)_\infty} \right\} \\ = \frac{1}{(cv; q)_\infty} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{(a; q)_n b^n q^{k(k-n)}}{(q; q)_k (q; q)_{n-k}} v^k D_q^{n-k} \left\{ \frac{1}{(csq^k, ctq^k, cuq^k, czq^k; q)_\infty} \right\} \\ = \frac{1}{(cv; q)_\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q)_{n+k} b^{n+k} q^{-nk} v^k}{(q; q)_n (q; q)_k} D_q^n \left\{ \frac{1}{(csq^k, ctq^k, cuq^k, czq^k; q)_\infty} \right\} \\ = \frac{1}{(cv; q)_\infty} \sum_{k=0}^{\infty} \frac{(a; q)_k (bv)^k}{(q; q)_k} \sum_{n=0}^{\infty} \frac{(aq^k; q)_n (bq^{-k})^n}{(q; q)_n} D_q^n \left\{ \frac{1}{(csq^k, ctq^k, cuq^k, czq^k; q)_\infty} \right\} \\ = \frac{1}{(cv; q)_\infty} \sum_{k=0}^{\infty} \frac{(a; q)_k (bv)^k}{(q; q)_k} T(aq^k, bq^{-k}; D_q) \left\{ \frac{1}{(csq^k, ctq^k, cuq^k, czq^k; q)_\infty} \right\} \\ = \frac{1}{(cv; q)_\infty} \sum_{k=0}^{\infty} \frac{(a; q)_k (bv)^k}{(q; q)_k} \frac{(absq^k; q)_\infty}{(bs, csq^k, ctq^k, cuq^k, czq^k; q)_\infty} \\ \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(aq^k, csq^k; q)_{i+j+l}}{(absq^k; q)_{i+j+l}} (ctq^k; q)_{j+l} (cuq^k; q)_j \frac{(bt)^i (bz)^j (bu)^l}{(q; q)_i (q; q)_j (q; q)_l}.$$

Which becomes the desired identity after using (1.1).

**Definition (2.2):** Let the  $q$  –shifted factorial and Gaussian polynomials defined as above, then:

$$Z_n(x; q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (x; q)_k. \tag{2.2}$$

where the polynomials (2.2) is a special case of  $q$  –bivariate polynomials  $\emptyset_n(a, x; q)$  given by Carlitz [5]. the polynomials (2.2) can be represented by the Cauchy operator as follows:

**Proposition (2.3):** Let the Cauchy operator  $T(a, b; D_q)$  and the polynomial  $Z_n(x; q)$  are defined as above, we have:

$$\lim_{c \rightarrow 1} T(x, 1; D_q) \{c^n\} = Z_n(x; q). \tag{2.3}$$

*Proof.*

By using the definition of the Cauchy operator and identity (1.5).

For example:

Take  $n = 1$  in proposition (2.3), get:

L.H.S:

$$\begin{aligned} &= \lim_{c \rightarrow 1} T(x, 1; D_q)\{c^1\} = \lim_{c \rightarrow 1} \sum_{k=0}^{\infty} \frac{(x; q)_k}{(q; q)_k} D^k_q \{c^1\} = \lim_{c \rightarrow 1} \sum_{k=0}^{\infty} \frac{(x; q)_k}{(q; q)_k} \frac{(q; q)_1}{(q; q)_{1-k}} c^{1-k} \\ &= \lim_{c \rightarrow 1} \sum_{k=0}^{\infty} \begin{bmatrix} 1 \\ k \end{bmatrix} (x; q)_k c^{1-k} = \lim_{c \rightarrow 1} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} (x; q)_0 c^1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (x; q)_1 c^0 \right) \\ &= \lim_{c \rightarrow 1} \left( \frac{(q; q)_1 c^1}{(q; q)_0 (q; q)_1} + \frac{(q; q)_1 (1-x)}{(q; q)_1 (q; q)_0} \right) = \lim_{c \rightarrow 1} (c + 1 - x) = 2 - x. \end{aligned}$$

R.H.S:

$$\begin{aligned} Z_1(x; q) &= \sum_{k=0}^1 \begin{bmatrix} 1 \\ k \end{bmatrix} (x; q)_k = \begin{bmatrix} 1 \\ 0 \end{bmatrix} (x; q)_0 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (x; q)_1 \\ &= \frac{(q; q)_1 \cdot 1}{(q; q)_0 (q; q)_1} + \frac{(q; q)_1 (1-x)}{(q; q)_1 (q; q)_0} = 1 + 1 - x = 2 - x. \end{aligned}$$

Now we derive the generating function after representing the polynomials  $Z_n(x; q)$  by (2.3) and using the identity (1.7) of the Cauchy operator.

**Theorem (2.4):** (The generating function of  $Z_n(x; q)$ ), we have:

$$\sum_{n=0}^{\infty} Z_n(x; q) \frac{t^n}{(q; q)_n} = \frac{(xt; q)_{\infty}}{[(t; q)_{\infty}]^2}. \tag{2.4}$$

where  $|t| < 1$ .

*Proof.*

$$\begin{aligned} \sum_{n=0}^{\infty} Z_n(x; q) \frac{t^n}{(q; q)_n} &= \sum_{n=0}^{\infty} \lim_{c \rightarrow 1} T(x, 1; D_q)\{c^n\} \frac{t^n}{(q; q)_n} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(ct)^n}{(q; q)_n} \right\} = \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{1}{(ct; q)_{\infty}} \right\} \\ &= \lim_{c \rightarrow 1} \left\{ \frac{(xt; q)_{\infty}}{(t, ct; q)_{\infty}} \right\} = \frac{(xt; q)_{\infty}}{[(t; q)_{\infty}]^2}. \end{aligned}$$

Here, we derive Mehler's formula by using identity (1.9) of the Cauchy operator, where we can represent the polynomials  $Z_n(x; q)$  by this operator according to (2.3) as  $\lim_{c \rightarrow 1} T(x, 1; D_q)\{c^n\}$  or the polynomials  $Z_n(y; q)$  as  $\lim_{c \rightarrow 1} T(y, 1; D_q)\{c^n\}$  to get the following result.

**Theorem (2.5):** (Mehler's formula of  $Z_n(x; q)$ ), we have:

$$\sum_{n=0}^{\infty} Z_n(x; q) Z_n(y; q) \frac{t^n}{(q; q)_n} = \frac{(xt, yt; q)_{\infty}}{[(t; q)_{\infty}]^3} {}_3\phi_2 \left( \begin{matrix} x, y, t \\ xt, yt \end{matrix}; q, t \right), \tag{2.5}$$

where  $|t| < 1$ .

*Proof.*

$$\begin{aligned} \sum_{n=0}^{\infty} Z_n(x; q)Z_n(y; q) \frac{t^n}{(q; q)_n} &= \sum_{n=0}^{\infty} \lim_{c \rightarrow 1} T(x, 1; D_q)\{c^n\} Z_n(y; q) \frac{t^n}{(q; q)_n} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \sum_{n=0}^{\infty} Z_n(y; q) \frac{(ct)^n}{(q; q)_n} \right\} = \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{(yct; q)_{\infty}}{(ct, ct; q)_{\infty}} \right\} \\ &= \lim_{c \rightarrow 1} \left\{ \frac{(xt, yct; q)_{\infty}}{(t, ct, ct; q)_{\infty}} {}_3\phi_2 \left( \begin{matrix} x, y, ct \\ xt, yct \end{matrix}; q, t \right) \right\} = \frac{(xt, yt; q)_{\infty}}{[(t; q)_{\infty}]^3} {}_3\phi_2 \left( \begin{matrix} x, y, t \\ xt, yt \end{matrix}; q, t \right). \end{aligned}$$

The Rogers formula of  $Z_n(x; q)$  will be given by using (2.3) and identity (1.8) of the Cauchy operator.

**Theorem (2.6):** (The Roger's formula of  $Z_n(x; q)$ ), we have:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m}(x; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(xt; q)_{\infty}}{(s; q)_{\infty} [(t; q)_{\infty}]^2} {}_2\phi_1 \left( \begin{matrix} x, t \\ xt \end{matrix}; q, s \right), \tag{2.6}$$

where  $\max \{ |t|, |s| \} < 1$ .

*Proof:*

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m}(x; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lim_{c \rightarrow 1} T(x, 1; D_q)\{c^{n+m}\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(ct)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(cs)^m}{(q; q)_m} \right\} = \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{1}{(cs, ct; q)_{\infty}} \right\} \\ &= \lim_{c \rightarrow 1} \left\{ \frac{(xt; q)_{\infty}}{(t, cs, ct; q)_{\infty}} {}_2\phi_1 \left( \begin{matrix} x, ct \\ xt \end{matrix}; q, s \right) \right\} = \frac{(xt; q)_{\infty}}{(s; q)_{\infty} [(t; q)_{\infty}]^2} {}_2\phi_1 \left( \begin{matrix} x, t \\ xt \end{matrix}; q, s \right). \end{aligned}$$

**An Extended Identities of  $Z_n(x; q)$  Polynomials**

In this section, we introduce three extended identities, the first is an extension of the generating function, the second is an extension of the Mehler's formula, and the third is an extension of the Rogers formula by using the roles (1.10), (1.11), and (1.12) of the Cauchy operator with our representation (2.3).

Firstly, we derive an extension of the generating function after representing the polynomials  $Z_n(x; q)$  by (2.3) and using the identity (1.10) of the Cauchy operator.

**Theorem (3.1):** (Extended generating function of  $Z_n(x; q)$ ), for  $k \in N$ , we have:

$$\sum_{n=0}^{\infty} Z_{n+k}(x; q) \frac{t^n}{(q; q)_n} = \frac{(xt; q)_{\infty}}{[(t; q)_{\infty}]^2} \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} \frac{(x, t; q)_j}{(xt; q)_j}. \tag{3.1}$$

where  $|t| < 1$ .

*Proof.*

$$\begin{aligned} \sum_{n=0}^{\infty} Z_{n+k}(x; q) \frac{t^n}{(q; q)_n} &= \sum_{n=0}^{\infty} \lim_{c \rightarrow 1} T(x, 1; D_q) \{c^{n+k}\} \frac{t^n}{(q; q)_n} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ c^k \sum_{n=0}^{\infty} \frac{(ct)^n}{(q; q)_n} \right\} = \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{c^k}{(ct; q)_{\infty}} \right\} \\ &= \lim_{c \rightarrow 1} \left\{ \frac{(xt; q)_{\infty}}{(t, ct; q)_{\infty}} \sum_{j=0}^k [k] \frac{(x, ct; q)_j}{(xt; q)_j} c^{k-j} \right\} = \frac{(xt; q)_{\infty}}{[(t; q)_{\infty}]^2} \sum_{j=0}^k [k] \frac{(x, t; q)_j}{(xt; q)_j}. \end{aligned}$$

Setting  $k = 0$  in (3.1), get the generating function (2.4) for the polynomials  $Z_n(x; q)$ .

Now we give an extension of Mehler's formula after representing the polynomials  $Z_n(x; q)$  by (2.3) and using the identity (1.12) of the Cauchy operator.

**Theorem (3.2):** (Extended Mehler's formula of  $Z_n(x; q)$ ), for  $k \in N$ , we have:

$$\sum_{n=0}^{\infty} Z_{n+k}(x; q) Z_n(y; q) \frac{t^n}{(q; q)_n} = \frac{(xt, yt; q)_{\infty}}{[(t; q)_{\infty}]^3} \sum_{l=0}^{\infty} \sum_{j=0}^k [k] \frac{(x, t; q)_{j+l} (t; q)_j (y; q)_l t^l}{(xt, yt; q)_{j+l} (q; q)_l} \quad (3.2)$$

where  $|t| < 1$ .

*Proof.*

$$\begin{aligned} \sum_{n=0}^{\infty} Z_{n+k}(x; q) Z_n(y; q) \frac{t^n}{(q; q)_n} &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ c^k \sum_{n=0}^{\infty} Z_n(y; q) \frac{(ct)^n}{(q; q)_n} \right\} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{c^k (yct; q)_{\infty}}{(ct, ct; q)_{\infty}} \right\} \\ &= \lim_{c \rightarrow 1} \left\{ \frac{(xt, yct; q)_{\infty}}{(t, ct, ct; q)_{\infty}} \sum_{l=0}^{\infty} \sum_{j=0}^k [k] \frac{(x, ct; q)_{j+l} (ct; q)_j (y; q)_l}{(xt, yct; q)_{j+l} (q; q)_l} c^{k-j} t^l \right\} \end{aligned}$$

The desired identity will be follows after substitute  $c \rightarrow 1$ .

Setting  $k = 0$  in (3.2), get Mehler's formula (2.5) for the polynomials  $Z_n(x; q)$ .

Finally, we introduce an extension of the Roger's formula by representing the polynomials  $Z_n(x; q)$  by the Cauchy operator according to (2.3) and using the identity (1.11) of the Cauchy operator.

**Theorem (3.3):** (Extended Roger's formula of  $Z_n(x; q)$ ), for  $k \in N$ , we have:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m+k}(x; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \frac{(xt; q)_{\infty}}{(s; q)_{\infty} [(t; q)_{\infty}]^2} \sum_{l=0}^{\infty} \sum_{j=0}^k [k] \frac{(x, t; q)_{j+l} (s; q)_j}{(xt; q)_{j+l} (q; q)_l} s^l \quad (3.3)$$

where  $\max \{ |t|, |s| \} < 1$ .

*Proof.*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m+k}(x; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lim_{c \rightarrow 1} T(x, 1; D_q) \{c^{n+m+k}\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m}$$

$$\begin{aligned}
 &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ c^k \sum_{n=0}^{\infty} \frac{(ct)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(cs)^m}{(q; q)_m} \right\} = \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{c^k}{(cs, ct; q)_{\infty}} \right\} \\
 &= \lim_{c \rightarrow 1} \left\{ \frac{(xt; q)_{\infty}}{(t, cs, ct; q)_{\infty}} \sum_{l=0}^{\infty} \sum_{j=0}^k [j] \frac{(x, ct; q)_{j+l} (cs; q)_j}{(xt; q)_{j+l} (q; q)_l} c^{k-j} s^l \right\} \\
 &= \frac{(xt; q)_{\infty}}{(s; q)_{\infty} [(t; q)_{\infty}]^2} \sum_{l=0}^{\infty} \sum_{j=0}^k [j] \frac{(x, t; q)_{j+l} (s; q)_j}{(xt; q)_{j+l} (q; q)_l} s^l.
 \end{aligned}$$

Setting  $k = 0$  in (3.3), get Roger's formula (2.6) for the polynomials  $Z_n(x; q)$ .

**Another Extended Identities of  $Z_n(x; q)$  Polynomial**

In this section, we continue in our technique to introduce some general extended identities involves a  ${}_3\phi_2$  sum and  ${}_2\phi_1$  sum in some cases.

**Theorem (4.1):** Let the polynomial  $Z_n(x; q)$  defined as in (2.2), for  $n, m \in N$ , we have:

$$\begin{aligned}
 &\sum_{k=0}^{\infty} Z_{m+k}(x; q) Z_{n+k}(y; q) \frac{t^k}{(q; q)_k} = (xt; q)_{\infty} \sum_{j=0}^m [j] (x; q)_j \sum_{l=0}^n [l] (y; q)_l \\
 &\times \frac{(ytq^l, xtq^{j+l}; q)_{\infty}}{(t, tq^l, tq^{j+l}; q)_{\infty}} {}_3\phi_2 \left( \begin{matrix} yq^l, & tq^l, & x \\ yq^l, & xtq^{j+l} \end{matrix}; q, tq^j \right).
 \end{aligned} \tag{4.1}$$

where  $|t| < 1$ .

*Proof.*

$$\begin{aligned}
 &\sum_{k=0}^{\infty} Z_{m+k}(x; q) Z_{n+k}(y; q) \frac{t^k}{(q; q)_k} \\
 &= \lim_{c \rightarrow 1} T(y, 1; D_q) \left\{ c^n \sum_{k=0}^{\infty} Z_{m+k}(x; q) \frac{(ct)^k}{(q; q)_k} \right\} \\
 &= \lim_{c \rightarrow 1} T(y, 1; D_q) \left\{ \frac{c^n (xct; q)_{\infty}}{(ct, ct; q)_{\infty}} \sum_{j=0}^m [j] \frac{(x, ct; q)_j}{(xct; q)_j} \right\} \\
 &= (xt; q)_{\infty} \sum_{j=0}^m [j] (x; q)_j \lim_{c \rightarrow 1} T(y, 1; D_q) \left\{ \frac{c^n (xctq^j; q)_{\infty}}{(ct, ctq^j; q)_{\infty}} \right\} \\
 &= (xt; q)_{\infty} \sum_{j=0}^m [j] (x; q)_j \lim_{c \rightarrow 1} \left\{ \sum_{l=0}^n [l] (y; q)_l \frac{c^{n-l} (ytq^l, xctq^{j+l}; q)_{\infty}}{(t, ctq^l, ctq^{j+l}; q)_{\infty}} \right. \\
 &\times \left. {}_3\phi_2 \left( \begin{matrix} yq^l, & ctq^l, & x \\ yq^l, & xctq^{j+l} \end{matrix}; q, tq^j \right) \right\}.
 \end{aligned}$$

Which get the required identity when  $c$  approach to 1.

**Theorem (4.2):** Let the polynomial  $Z_n(x; q)$  defined as in (2.2), we have:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} Z_{n+m+k}(x; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{v^k}{(q; q)_k}$$

$$= \frac{1}{(v; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(x; q)_j (xtq^j; q)_{\infty} v^j}{(t, sq^j, tq^j; q)_{\infty} (q; q)_j} {}_2\phi_1 \left( \begin{matrix} xq^j, tq^j \\ xtq^j \end{matrix}; q, s \right). \tag{4.2}$$

where  $\max \{ |t|, |s|, |v| \} < 1$ .

*Proof.*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} Z_{n+m+k}(x; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{v^k}{(q; q)_k}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \lim_{c \rightarrow 1} T(x, 1; D_q) \{c^{n+m+k}\} \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \frac{v^k}{(q; q)_k}$$

$$= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(ct)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(cs)^m}{(q; q)_m} \sum_{k=0}^{\infty} \frac{(cv)^k}{(q; q)_k} \right\}$$

$$= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{1}{(cs, ct, cv; q)_{\infty}} \right\}$$

$$= \lim_{c \rightarrow 1} \left\{ \frac{1}{(cv; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(x; q)_j}{(t, csq^j, ctq^j; q)_{\infty}} \frac{(xtq^j; q)_{\infty} v^j}{(q; q)_j} {}_2\phi_1 \left( \begin{matrix} xq^j, ctq^j \\ xtq^j \end{matrix}; q, s \right) \right\}$$

$$= \frac{1}{(v; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(x; q)_j (xtq^j; q)_{\infty} v^j}{(t, sq^j, tq^j; q)_{\infty} (q; q)_j} {}_2\phi_1 \left( \begin{matrix} xq^j, tq^j \\ xtq^j \end{matrix}; q, s \right).$$

**Theorem (4.3):** Let the polynomial  $Z_n(x; q)$  defined as in (2.2), we have:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Z_{n+m+k+l}(x; q) \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \frac{u^k}{(q; q)_k} \frac{z^l}{(q; q)_l}$$

$$= \frac{(xs; q)_{\infty}}{[(s; q)_{\infty}]^2 (t, u, z; q)_{\infty}} \sum_{\alpha, \beta, \gamma=0}^{\infty} \frac{(x, s; q)_{\alpha+\beta+\gamma} (t; q)_{\beta+\gamma} (u; q)_{\beta}}{(xs; q)_{\alpha+\beta+\gamma}} \frac{t^{\alpha} z^{\beta} u^{\gamma}}{(q; q)_{\alpha} (q; q)_{\beta} (q; q)_{\gamma}}.$$

(4.3)

where  $\max \{ |s|, |t|, |u|, |z| \} < 1$ .

*Proof.*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} Z_{n+m+k+l}(x; q) \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \frac{u^k}{(q; q)_k} \frac{z^l}{(q; q)_l}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \lim_{c \rightarrow 1} T(x, 1; D_q) \{c^{n+m+k+l}\} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \frac{u^k}{(q; q)_k} \frac{z^l}{(q; q)_l}$$

$$= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(cs)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(ct)^m}{(q; q)_m} \sum_{k=0}^{\infty} \frac{(cu)^k}{(q; q)_k} \sum_{l=0}^{\infty} \frac{(cz)^l}{(q; q)_l} \right\}$$

$$= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{1}{(cs, ct, cu, cz; q)_\infty} \right\}.$$

By setting  $v = w = 0$ ,  $a = x$  and  $b = 1$  in (1.14) or setting  $v = 0$  in (2.1) of the Cauchy operator, then  $c \rightarrow 1$ , the desired identity follows.

**Theorem (4.4):** Let the polynomial  $Z_n(x; q)$  defined as in (2.2), we have:

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} Z_{n+m+k+l+p}(x; q) \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \frac{u^k}{(q; q)_k} \frac{z^l}{(q; q)_l} \frac{v^p}{(q; q)_p} \\ &= \frac{(xs; q)_\infty}{[(s; q)_\infty]^2 (t, u, z, v; q)_\infty} \sum_{\alpha, \beta, \gamma, \sigma=0}^{\infty} \frac{(x, s; q)_{\alpha+\beta+\gamma+\sigma} (t; q)_{\alpha+\gamma+\sigma} (u; q)_{\alpha+\gamma} (z; q)_\alpha v^\alpha t^\beta z^\gamma u^\sigma}{(xs; q)_{\alpha+\beta+\gamma+\sigma} (q; q)_\alpha (q; q)_\beta (q; q)_\gamma (q; q)_\sigma} \end{aligned} \tag{4.4}$$

where  $\max\{|s|, |t|, |u|, |z|, |v|\} < 1$ .

*Proof.*

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} Z_{n+m+k+l+p}(x; q) \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \frac{u^k}{(q; q)_k} \frac{z^l}{(q; q)_l} \frac{v^p}{(q; q)_p} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \lim_{c \rightarrow 1} T(x, 1; D_q) \{c^{n+m+k+l+p}\} \frac{s^n}{(q; q)_n} \frac{t^m}{(q; q)_m} \frac{u^k}{(q; q)_k} \frac{z^l}{(q; q)_l} \frac{v^p}{(q; q)_p} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(cs)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(ct)^m}{(q; q)_m} \sum_{k=0}^{\infty} \frac{(cu)^k}{(q; q)_k} \sum_{l=0}^{\infty} \frac{(cz)^l}{(q; q)_l} \sum_{p=0}^{\infty} \frac{(cv)^p}{(q; q)_p} \right\} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{1}{(cs, ct, cu, cz, cv; q)_\infty} \right\}. \end{aligned}$$

Which equals the required identity by using (2.1) and the parameter  $c$  approach to 1.

**Theorem (4.5):** Let the polynomial  $Z_n(x; q)$  defined as in (2.2), we have:

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m}(x; q) Z_n(y; q) Z_m(z; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \frac{(xt, yt, zs; q)_\infty}{[(t; q)_\infty]^3 [(s; q)_\infty]^2} \sum_{j, k, l=0}^{\infty} \frac{(x, t; q)_{j+k+l} (t; q)_{k+l}}{(xt, yt; q)_{j+k+l}} \frac{(s, z; q)_k (y; q)_j t^j s^{k+l}}{(zs; q)_k (q; q)_j (q; q)_k (q; q)_l} \end{aligned} \tag{4.5}$$

where  $\max\{|t|, |s|\} < 1$ .

*Proof.*

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_{n+m}(x; q) Z_n(y; q) Z_m(z; q) \frac{t^n}{(q; q)_n} \frac{s^m}{(q; q)_m} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \sum_{n=0}^{\infty} Z_n(y; q) \frac{(ct)^n}{(q; q)_n} \sum_{m=0}^{\infty} Z_m(z; q) \frac{(cs)^m}{(q; q)_m} \right\} \\ &= \lim_{c \rightarrow 1} T(x, 1; D_q) \left\{ \frac{(yct; q)_\infty (yct; q)_\infty}{[(ct; q)_\infty]^2 [(cs; q)_\infty]^2} \right\}. \end{aligned}$$

The desired identity will be followed by applying the role (1.14) of the Cauchy operator with setting  $a = x, b = 1, v = yt, w = zs, u = z = s$  and  $s = t$ , then  $c \rightarrow 1$ . Finally we introduce the following identity which is an extension of the previous identity (4.5) of  $Z_n(x; q)$  polynomial.

**Theorem (4.6):** Let the polynomial  $Z_n(x; q)$  defined as in (2.2), we have:

$$\sum_{n,m,k,j=0}^{\infty} Z_{n+m}(x; q)Z_{k+j}(y; q) Z_n(z; q)Z_m(w; q)Z_k(f; q)Z_j(g; q) \frac{t^n s^m c^k r^j}{(q; q)_n (q; q)_m (q; q)_k (q; q)_j}$$

$$= \frac{(xt, zt, ws; q)_{\infty}}{[(t; q)_{\infty}]^3 [(s; q)_{\infty}]^2} \sum_{a,b,i=0}^{\infty} \frac{(x, t; q)_{a+b+i} (t; q)_{b+i}}{(xt, zt; q)_{a+b+i}} \frac{(s, w; q)_b (z; q)_a t^a s^{b+i}}{(ws; q)_b (q; q)_a (q; q)_b (q; q)_i}$$

$$\times \frac{(yc, fc, gr; q)_{\infty}}{[(c; q)_{\infty}]^3 [(r; q)_{\infty}]^2} \sum_{\alpha,\beta,\gamma=0}^{\infty} \frac{(y, c; q)_{\alpha+\beta+\gamma} (c; q)_{\beta+\gamma}}{(yc, fc; q)_{\alpha+\beta+\gamma}} \frac{(r, g; q)_{\beta} (f; q)_{\gamma} c^{\alpha} r^{\beta+\gamma}}{(gr; q)_{\beta} (q; q)_{\alpha} (q; q)_{\beta} (q; q)_{\gamma}}$$

where  $\max\{|t|, |s|, |c|, |r|\} < 1$ . (4.6)

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