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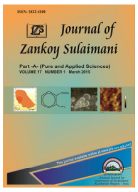

Some Properties of Sp-Closed Spaces and Sp-Closed Relative to Spaces

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Abstract

The purpose of this paper is to introduce two types of sets called S_p- **θ**- open and S_p-regular set in topological spaces and study more properties of Sp-closed spaces, S_p-closed relative to spaces interms of these two sets.

Key Words:
Sp- ***θ***- open set
Sp-regular set
Sp-closed space
Sp-closed relative to space

I. Introduction

Throughout this paper by a space X we mean a topological space on which no separation axioms are assumed unless explicitly stated. The closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. The notion of S_p-open sets was introduce in 2007 by Khalaf A. B. and Shareef H. A. [10]. In 2012 Khalaf A. B. and Shareef H. A. defined S_p-closed spaces by using S_p-open sets and it's S_p-closure of this set [5] while in 2013 Shareef H. A. introduced the notion of S_p-closed subspace and S_p-closed relative to spaces [11]. The $S_p\mathrm{cl}(A)$ and $S_p\mathrm{int}(A)$ denote the S_p-closure and Sp-interior of A, while $scl(A)$ denote the semi-closure of A.

II. Preliminaries:

Definition 2.1: [10] A subset A of a space X is called S_p-open set if it is semi-open set and for each $x \in A$, there exists a pre-closed set U such that $x \in U \subseteq A$. And the complement of an S_p-open sets is called S_p-closed.

Definition 2.2: [5] A space X is said to be S_p-closed if for every S_p-open cover $\{V_\alpha : \alpha \in \Delta\}$ of X there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha\in\Delta_0} S_p\mathrm{cl}(V_\alpha)$.

Lemma 2.3: [10] A subset A of a space X is S_p-closed set if and only if $A = S_p\mathrm{cl}(A)$.

Definition 2.4: [11] A subset A of a space X is called S_p-closed relative to X if for every cover $\{V_\alpha : \alpha \in \Delta\}$ of A by S_p-open sets of X, there exists a finite subset Δ_0 of Δ such that $A \subseteq \bigcup_{\alpha\in\Delta_0} S_p\mathrm{cl}(V_\alpha)$.

Definition 2.5: [3] A space X is said to be locally indiscrete if every open subset of X is closed.

Definition 2.6: A subset A of a space X is said to be semi-open [6] (resp. preopen [8], regular closed and regular open [13]) set if $A \subseteq \mathrm{clint}(A)$ (resp. $A \subseteq \mathrm{intcl}(A)$, $A = \mathrm{clint}(A)$ and $A = \mathrm{intcl}(A)$). And A is called semi-regular set if A is both semi-open and semi-closed set in X [7].

The family of all semi-open, preopen, regular open, regular closed and S_p-open sets of a space X denoted by $SO(X)$, $PO(X)$, $RO(X)$, $RC(X)$ and $S_pO(X)$.

Lemma 2.7: [2] A subset A of a space X is semi-closed if and only if $A = \mathrm{scl}(A)$.

Theorem 2.8: [3] A space X X is locally indiscrete if and only if every subset of X is a preopen set.

Theorem 2.9: [1] Let A be a subset of a space X. Then A is preopen set if and only if $\mathrm{scl}(A) = \mathrm{intcl}(A)$.

Theorem 2.10: [2] A subset A of space X is semi-closed if and only if $\mathrm{intcl}(A) \subseteq A$.

Theorem 2.11: [10] If a space X is locally indiscrete, then every open set is S_p-open.

Definition 2.12: [9] A space X is said to be extremally disconnected space if the closure of every open set is open.

Lemma 2.13: [10] If a space X is extremally disconnected, then every S_p-open subset of X is preopen.

Proposition 2.14: [11] Let Y be a regular closed subspace of a space X and $A \subseteq Y$. If A is S_p-closed relative to X, then A is S_p-closed relative to Y.

Definition 2.15: [10] A space X is said to be p-Alexandroff if any intersection of preopen sets is preopen set. Or equivalently any union of preclosed sets is preclosed.

Theorem 2.16: [3] A space X is extremally disconnected if and only if every semi-open subset of X is preopen.

Lemma 2.17: [11] For any preopen subset A of a space X, $\mathrm{scl}(A) = \mathrm{int}(\mathrm{cl}(A)) = S_p\mathrm{cl}(A)$.

Proposition 2.18: [4] A space X is extremally disconnected space if and only if $RO(X) = RC(X)$.

Definition 2.19: [7] A space X is said to be an s-closed space if for every semi-open cover $\{V_\alpha : \alpha \in \Delta\}$ of X, there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha \in \Delta_0} \mathrm{scl}(V_\alpha)$.

Proposition 2.20: [5] Every s-closed space is S_p-closed.

Lemma 2.21: [10] Let A and B be subsets of a space X. If $A \subseteq B$, then $S_p\mathrm{cl}(A) \subseteq S_p\mathrm{cl}(B)$.

Definition 2.22: [12] A filter base on a space X is a non-empty family β of subsets of X with the following conditions:

1. $\emptyset \notin \beta$.
2. If $A, B \in \beta$, , then there exists $G \in \beta$ such that $G \subseteq A \cap B$.

Definition 2.23: [5] A filter base β on a space X is S_p-θ- accumulates to a point $x \in X$ if for every S_p-open set U containing x and every $B \in \beta$, $B \cap S_p\mathrm{cl}(U) \neq \emptyset$.

Proposition 2.24: [5] A space X is an S_p-closed if and only if each filter base β in X is S_p-θ-accumulates to a point $x_0 \in X$.

III. Main Results:

Definition 3.1: Let A be a subset of X. Then A is called S_p- θ-open set if for each $x \in A$, there exist an S_p-open set U of X such that $x \in U \subseteq S_p\mathrm{cl}(U) \subseteq A$. The family of all S_p-θ- open sets is denoted by $S_p\theta O(X)$.

Definition 3.2: A subset A of a space X is called S_p- regular set if it is both S_p- open and S_p- closed set in X. The family of all S_p- regular set is denoted by $S_pR(X)$.

From the above definitions it is clear that every S_p-θ-open set is S_p-open set and every S_p-regular set is S_p-open set, but the converse of them are not true as: If $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, then the family of S_p-open sets with respect to τ is $S_pO(X) = \{\phi, X, \{a, c\}, \{b, c\}\}$. Here $\{a, c\}$ is S_p-open set but not S_p-θ-open and nor S_p-regular set.

Theorem 3.3: If a space X is S_p-closed, then every proper S_p-regular subset of X is S_p-closed relative to X.

Proof: Let G be any proper S_p-regular subset of X and let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of G by S_p-open subsets of X. Then clearly $X \backslash G$ is also S_p-regular set in X this implies that $\{V_\alpha : \alpha \in \Delta\} \cup (X \backslash G)$ form an S_p-open cover of X, but since X is S_p- closed space so there exists a finite subfamily Δ_0 of Δ such that

$X = [\bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)] \cup \mathrm{S_p cl}(X\backslash G) = [\bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)] \cup (X\backslash G)$ by Lemma 2.3, and then $G \subseteq \bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)$ this implies that by Definition 2.4 G is $\mathrm{S_p}$-closed relative to X.

Theorem 3.4: If every proper $\mathrm{S_p}$-regular set in a locally indiscrete space X is $\mathrm{S_p}$- closed relative to X, then X is $\mathrm{S_p}$-closed space.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be any $\mathrm{S_p}$- open cover of X and let $G = \mathrm{S_p cl}(V_\beta)$ for a $\beta \in \Delta$. Then G is $\mathrm{S_p}$-closed set in X and since X is locally indiscrete space so by Theorem 2.8 G is preopen set, and then by Theorem 2.9 $scl\, G = intcl(G)$. But since every $\mathrm{S_p}$-open set is semi-open set so $scl(G) \subseteq \mathrm{S_p cl}(G) = G$ implies that $intcl(G) \subseteq G$ and then by theorem in Theorem 2.10 G is semi-closed, and then $scl(G) = G = intcl(G)$ implies that G is open set. Since X is locally indiscrete space so G is closed, and then G is preclosed set. Thus G is $\mathrm{S_p}$-open set implies that G is $\mathrm{S_p}$-regular. Now $X\backslash G$ is also $\mathrm{S_p}$-regular set and $X\backslash G \subseteq \bigcup_{\alpha\in\Delta} V_\alpha$ and since $X\backslash G$ is $\mathrm{S_p}$-closed relative to X so there exists a finite subset Δ_0 of Δ such that $X\backslash G \subseteq \bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)$ implies that $X = (\bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)) \cup \mathrm{S_p cl}(V_\beta)$. Hence X is $\mathrm{S_p}$-closed space.

Corollary 3.5: Let X be a locally indiscrete space. Then X is $\mathrm{S_p}$-closed space if and only if every proper $\mathrm{S_p}$-regular subset of X is $\mathrm{S_p}$-closed relative to X.

Proof: Follows from Theorem 3.3 and Theorem 3.4.

Theorem 3.6: Let A be any subset of a space X, then the following statements are true:

1. If A is $\mathrm{S_p}$-closed relative to X, then every cover of A by $\mathrm{S_p}$-θ-open set of X has a finite subcover.
2. If A is $\mathrm{S_p}$-closed relative to X, then every cover of A by $\mathrm{S_p}$-regular set of X has a finite subcover.

Proof:

1. Let A be $\mathrm{S_p}$-closed relative to X and $\{V_\alpha : \alpha \in \Delta\}$ be any cover of A by $\mathrm{S_p}$-θ-open sets in X. Then $A \subseteq \bigcup_{\alpha\in\Delta} V_\alpha$ implies that for each $x \in A$, there exists $\alpha_x \in \Delta$ such that $x \in V_{\alpha_x}$. But $V_{\alpha_x} \in \mathrm{S_p}\theta O(X)$ so by Definition 3.1 there exists an $\mathrm{S_p}$-open set U_x of X such that $x \in U_x \subseteq \mathrm{S_p cl}(U_x) \subseteq V_{\alpha_x}$ this implies that $A \subseteq \bigcup_{x\in A} U_x$, and then the family $\{U_x : x \in A\}$ is a cover of A by $\mathrm{S_p}$-open sets in X, but A is $\mathrm{S_p}$-closed relative to X so there exists a finite points $x_1, x_2, x_3, \dots, x_n$ in A such that $A \subseteq \bigcup_{i=1}^{n} \mathrm{S_p cl}(U_{x_i}) \subseteq \bigcup_{i=1}^{n} V_{\alpha_{x_i}}$. Thus every cover of A by $\mathrm{S_p}$-θ-open set of X has a finite subcover.
2. Let A be $\mathrm{S_p}$-closed relative to X and $\{V_\alpha : \alpha \in \Delta\}$ be any cover of A by $\mathrm{S_p}$-regular sets in X. Then by Definition 3.2 $\{V_\alpha : \alpha \in \Delta\}$ is a cover of A by $\mathrm{S_p}$-open sets in X and since A is $\mathrm{S_p}$-closed relative to X so there exists a finite subset Δ_0 of Δ such that $A \subseteq \bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha) = \bigcup_{\alpha\in\Delta_0} V_\alpha$ since by Lemma 2.3 $\mathrm{S_p cl}(V_\alpha) = V_\alpha$ for each $\alpha \in \Delta$. Thus every cover of A by $\mathrm{S_p}$-regular set of X has a finite subcover.

Lemma 3.7: Every $\mathrm{S_p}$- regular set in a space X is $\mathrm{S_p}$- θ-open set.

Proof: Let A be any $\mathrm{S_p}$-regular set in X. Then A is both $\mathrm{S_p}$-open and $\mathrm{S_p}$-closed set, and then for each $x \in A$ there exists an $\mathrm{S_p}$-open set which is A it self such that $x \in A = \mathrm{S_p cl}(A) \subseteq A$. Thus A is $\mathrm{S_p}$-θ- open set in X.

The converse of [Lemma 3.7] is not true in general as an example:

In the usual topological space (R, τ_U), $R - \{0\}$ is $\mathrm{S_p}$-θ-open set but not $\mathrm{S_p}$-regular set in R.

Proposition 3.8: For a locally indiscrete space the following statements are equivalent:

1. X is $\mathrm{S_p}$-closed space.
2. Every $\mathrm{S_p}$- θ -open cover of X has a finite subcover.
3. Every $\mathrm{S_p}$- regular cover of X has a finite subcover.

Proof: (1) → (2)

Follow from Theorem 3.6 (1).

(2) → (3)

Let (2) be satisfied and let $\{V_\alpha : \alpha \in \Delta\}$ be any $\mathrm{S_p}$-regular cover of X. Then by Lemma 3.7 $\{V_\alpha : \alpha \in \Delta\}$ is $\mathrm{S_p}$-θ-open cover of X and then by (2) there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha\in\Delta_0} V_\alpha$. Thus (3) is satisfied.

(3) → (1)

Let $\{V_\alpha : \alpha \in \Delta\}$ be any $\mathrm{S_p}$-open cover of X. Then $X = \bigcup_{\alpha\in\Delta} V_\alpha$ implies that $X = \bigcup_{\alpha\in\Delta} \mathrm{S_p cl}(V_\alpha)$. Since X is locally indiscrete space so for each $\alpha \in \Delta$, $\mathrm{scl}(V_\alpha) = \mathrm{S_p cl}(V_\alpha)$ also by Theorem 2.8 V_α is preopen sets for each $\alpha \in \Delta$, and then by Theorem 2.9 $scl(V_\alpha) = intcl(V_\alpha)$ implies that by Theorem 2.11 $\mathrm{S_p cl}(V_\alpha)$ is $\mathrm{S_p}$-open set. Thus for each $\alpha \in \Delta$, $\mathrm{S_p cl}(V_\alpha)$ is $\mathrm{S_p}$-regular set in X, then the family $\{\mathrm{S_p cl}(V_\alpha) : \alpha \in \Delta\}$ is an $\mathrm{S_p}$-regular cover of X and by (3) there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)$ implies that X is $\mathrm{S_p}$-closed space.

Proposition 3.9: Let A and B be subsets of a space X such that $A \cap B \neq \phi$. If A is $\mathrm{S_p}$-closed relative to X and B is $\mathrm{S_p}$-regular set in X, then $A \cap B$ is $\mathrm{S_p}$-closed relative to X.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $A \cap B$ by $\mathrm{S_p}$-open sets in X. Then $A \cap B \subseteq \bigcup_{\alpha\in\Delta} V_\alpha$, and since B is $\mathrm{S_p}$-regular set in X implies that $X\backslash B$ is also $\mathrm{S_p}$-regular in X and then $X\backslash B$ is $\mathrm{S_p}$-open set in X implies that $\{V_\alpha : \alpha \in \Delta\} \cup (X\backslash B)$ is a cover of A by $\mathrm{S_p}$-open sets in X. Since A is $\mathrm{S_p}$-closed relative to X, then there exists a finite subset Δ_0 of Δ such that $A \subseteq [\bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)] \cup (X\backslash B)$ this implies that $A \cap B \subseteq \bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)$. Thus $A \cap B$ is $\mathrm{S_p}$-closed relative to X.

Corollary 3.10: If B is an $\mathrm{S_p}$-regular set in an $\mathrm{S_p}$-closed space X, then B is $\mathrm{S_p}$-closed relative to X.

Proof: Follows from Proposition 3.9.

Corollary 3.11: If A and B are $\mathrm{S_p}$-regular subsets of an $\mathrm{S_p}$-closed space X such that $A \cap B \neq \phi$, then $A \cap B$ is $\mathrm{S_p}$-closed relative to X.

Proof: Follows from Proposition 3.9 and Corollary 3.10.

Lemma 3.12: If a space X is extremally disconnected space, then every $\mathrm{S_p}$-regular set in X is regular closed.

Proof: Follows from Lemma 2.13.

Corollary 3.13: Let A and B be subsets of an extremally disconnected space X such that $A \subseteq B$ and $B \in \mathrm{S_p R}(X)$. If A is $\mathrm{S_p}$-closed relative to X, then A is $\mathrm{S_p}$-closed relative to B.

Proof: Let A be $\mathrm{S_p}$-closed relative to X. Since X is extremally disconnected space and $B \in \mathrm{S_p\,R}(X)$, then by **Lemma 3.12:** $B \in RC(X)$ this implies that by Proposition 2.14 A is $\mathrm{S_p}$-closed relative to B.

Corollary 3.14: Let A be $\mathrm{S_p}$-closed relative to an extremally disconnected space X. Then for each $B \in S_pR(X)$, $A \cap B$ is $\mathrm{S_p}$-closed relative to B.

Proof: Since B is $\mathrm{S_p}$-regular set in X, so by Theorem 3.9 $A \cap B$ is $\mathrm{S_p}$-closed relative to X. But $A \cap B \subseteq B$ and X is extremally disconnected space, then by Corollary 3.13 $A \cap B$ is $\mathrm{S_p}$-closed relative to B.

Corollary 3.15: If B is an $\mathrm{S_p}$-regular set in a space X and A is $\mathrm{S_p}$-closed relative to X such that $B \subseteq A$, then B is $\mathrm{S_p}$-closed relative to X.

Proof: follows from Theorem 3.9.

Proposition 3.16: Let X be a locally indiscrete space and $A \in \mathrm{S_p C}(X)$. If A is $\mathrm{S_p}$-closed relative to X, then $\mathrm{S_p int}(A)$ is $\mathrm{S_p}$-closed relative to X.

Proof: Let $A \in \mathrm{S_p C}(X)$ and be $\mathrm{S_p}$-closed relative to X. Then by Lemma 2.3 $A = \mathrm{S_p cl}(A)$ and since X is locally indiscrete space, so by Theorem 2.8 A is preopen and by Theorem 2.9 $scl(A) = intcl(A)$. But A is semi-closed set in X implies that by Lemma 2.7 $A = scl(A)$ and then A is open set in X this implies that A is semi-open set also by Definition 2.5 A is closed, then A is preclosed in X. Therefore; A is $\mathrm{S_p}$-open set, and then A is $\mathrm{S_p}$-regular set in X this implies that $A = \mathrm{S_p int}(A) \in \mathrm{S_p R}(X)$, but A is $\mathrm{S_p}$-closed relative X so by Corollary 3.15 $\mathrm{S_p int}(A)$ is $\mathrm{S_p}$-closed relative X.

Lemma 3.17: If a space X is extremally disconnected and p-Alexandroff, then for any subset A of X, $\mathrm{S_p cl}(A) \in \mathrm{S_p R}(X)$

Proof: Let A be a subset of extremally disconnected and p-Alexandroff space X. Then $\mathrm{S_p cl}(A)$ is $\mathrm{S_p}$-closed in X. Now let $\mathrm{S_p cl}(A) = G$ implies that G is semi-closed and is the intersection of preopen set and since X is extremally disconnected space, then by Theorem 2.16 G is preclosed in X also since X is p-Alexandroff space so G is also preopen implies that by Lemma 2.17 $G = \mathrm{S_p cl}(G) = scl(G) = intcl(G)$. Then G is semi-open and since X is extremally disconnected space so $G \in RC(X)$ this implies that G is $\mathrm{S_p}$-open set. Hence $G = \mathrm{S_p cl}(A)$ is $\mathrm{S_p}$-regular set.

Theorem 3.18: Let X be p-Alexandroff and extremally disconnected space. Then X is $\mathrm{S_p}$-closed space if and only if every $\mathrm{S_p}$-regular cover of X has a finite subcover.

Proof: Let X be $\mathrm{S_p}$-closed space. Then by Theorem 3.6 (2) every $\mathrm{S_p}$-regular cover of X has a finite subcover.

Conversely: Let $\{V_\alpha : \alpha \in \Delta\}$ be any $\mathrm{S_p}$-open cover of X. Then $X = \bigcup_{\alpha\in\Delta} V_\alpha$ implies that $X = \bigcup_{\alpha\in\Delta} \mathrm{S_p cl}(V_\alpha)$ and since X is p-Alexandroff and extremally disconnected space, so by Lemma 3.17 for each $\alpha \in \Delta$, $\mathrm{S_p cl}(V_\alpha) \in \mathrm{S_pR}(X)$ this implies that by hypothesis there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha\in\Delta_0} \mathrm{S_p cl}(V_\alpha)$. Thus X is $\mathrm{S_p}$-closed space.

Lemma 3.19: If a space X is extremally disconnected space, then $\mathrm{S_p}$-regular set and semi-regular set are identical.

Proof: Clearly every $\mathrm{S_p}$-regular set is semi-regular and since X is extremally disconnected space, so by Theorem 2.16 every semi-regular set is $\mathrm{S_p}$-regular.

Proposition 3.20: If a space X is extremally disconnected and p-Alexandroff, then s-closed and $\mathrm{S_p}$-closed spaces are identical.

Proof: Follows from Lemma 3.19 and Proposition 2.20.

Definition 3.21: A point $x \in X$ is said to be $\mathrm{S_p}$-θ-adherent point of a subset A of X if for every $\mathrm{S_p}$-open set U containing x, $\mathrm{S_p cl}(U) \cap A \neq \phi$. The set of all $\mathrm{S_p}$-θ-adherent points of A is called $\mathrm{S_p}$-θ-closure of A and it is denoted by $\mathrm{S_p}\text{-}\mathrm{cl}_\theta(A)$. Clearly A is called $\mathrm{S_p}$-θ-closed if and only if $\mathrm{S_p}\text{-}\mathrm{cl}_\theta(A) = A$.

Proposition 3.22: Let X be an $\mathrm{S_p}$-closed space. Then for every family of $\mathrm{S_p}$-regular sets $\{V_\alpha : \alpha \in \Delta\}$ such that $\bigcap_{\alpha\in\Delta} V_\alpha = \phi$, there exists a finite subset Δ_0 of Δ such that $\bigcap_{\alpha\in\Delta_0} V_\alpha = \phi$.

Proof: Let $\{V_\alpha : \alpha \in \Delta\}$ be any family of $\mathrm{S_p}$-regular sets of X such that $\bigcap_{\alpha\in\Delta} V_\alpha = \phi$. Then $X = \bigcup_{\alpha\in\Delta} V_\alpha^c$, but $V_\alpha^c \in \mathrm{S_pR}(X)$ for each $\alpha \in \Delta$ this implies that $\{V_\alpha^c : \alpha \in \Delta\}$ is an $\mathrm{S_p}$-regular cover of X and since X is $\mathrm{S_p}$-closed so by Theorem 3.6 there exists a finite subset Δ_0 of Δ such that $X = \bigcup_{\alpha\in\Delta_0} V_\alpha^c$ this implies that $\bigcap_{\alpha\in\Delta_0} V_\alpha = \phi$.

Corollary 3.23: If a space X is locally indiscrete or p-Alexandroff and extremally disconnected, then X is $\mathrm{S_p}$-closed if and only if for every family of $\mathrm{S_p}$-regular sets $\{V_\alpha : \alpha \in \Delta\}$ such that $\bigcap_{\alpha\in\Delta} V_\alpha = \phi$, there exists a finite subset Δ_0 of Δ such that $\bigcap_{\alpha\in\Delta_0} V_\alpha = \phi$.

Proof: Follows from Proposition 3.22, Proposition 3.8 and Theorem 3.18.

Proposition 3.24: If a space X is $\mathrm{S_p}$-closed, then each family of $\mathrm{S_p}$-θ-closed subsets of X with finite intersection property has a non-void intersection.

Proof: Let X be $\mathrm{S_p}$-closed space and $\{F_\alpha : \alpha \in \Delta\}$ be any family of $\mathrm{S_p}$-θ-closed subsets of X which has a finite intersection property. To show $\bigcap_{\alpha\in\Delta} F_\alpha \neq \phi$, let β be the collection of all finite intersections of members of $\{F_\alpha : \alpha \in \Delta\}$. Then clearly β form a filter base on X, so by Proposition 2.24 β has an $\mathrm{S_p}$-θ-accumulation point say $x_0 \in X$ implies that for every $\mathrm{S_p}$-open set U containing x_0 and every $H \in \beta$, $\mathrm{S_p cl}(U) \cap H \neq \phi$ and then $x_0 \in \mathrm{S_p}\text{-}\mathrm{cl}_\theta(F_\alpha)$ for all $\alpha \in \Delta$, but F_α is $\mathrm{S_p}$-θ-closed set in X for each $\alpha \in \Delta$ this implies that $x \in F_\alpha$ for each $\alpha \in \Delta$. And then $x_0 \in \bigcap_{\alpha\in\Delta} F_\alpha$ this implies that $\bigcap_{\alpha\in\Delta} F_\alpha \neq \phi$.

Lemma 3.25: Let X be any space and A be a subset of X. If $A \in \mathrm{S_pO}(X)$, then $\mathrm{S_p cl}(A) = \mathrm{S_p}\text{-}\mathrm{cl}_\theta(A)$.

Proof: Let $x \in \mathrm{S_p cl}(A)$. Then for every $\mathrm{S_p}$-open set U which containing x, we have $U \cap A \neq \phi$ this implies that $\mathrm{S_p cl}(U) \cap A \neq \phi$ and then $x \in \mathrm{S_p}\text{-}\mathrm{cl}_\theta(A)$, thus $\mathrm{S_p cl}(A) \subseteq \mathrm{S_p}\text{-}\mathrm{cl}_\theta(A)$.

Now let $x \in \mathrm{S_p}\text{-}\mathrm{cl}_\theta(A)$. To show $x \in \mathrm{S_p cl}(A)$, if possible suppose that $x \notin \mathrm{S_p cl}(A)$ this implies that there exists an $\mathrm{S_p}$-open set U containing x such that $U \cap A = \phi$, then $U \subseteq X \backslash A$ and then by Lemma 2.21 $\mathrm{S_p cl}(U) \subseteq \mathrm{S_p cl}(X\backslash A) = X\backslash A$ implies that $\mathrm{S_p cl}(U) \cap A = \phi$, then $x \notin \mathrm{S_p}\text{-}\mathrm{cl}_\theta(A)$ which is a contradiction, thus $x \in \mathrm{S_p cl}(A)$ this implies that $\mathrm{S_p}\text{-}\mathrm{cl}_\theta(A) \subseteq \mathrm{S_p cl}(A)$. Hence $\mathrm{S_p}\text{-}\mathrm{cl}_\theta(A) = \mathrm{S_p cl}(A)$.

Proposition 3.26: Let X be a locally indiscrete space. Then X is an $\mathrm{S_p}$-closed space if and only if every family of $\mathrm{S_p}$-θ-closed subsets of X with finite intersection property, has non-void intersections.

Proof: Let X be $\mathrm{S_p}$-closed space. Then by Proposition 3.24 then every family of $\mathrm{S_p}$-θ-closed subsets of X with finite intersection property, has non-void intersection.

Conversely: let the hypothesis be satisfied, to show X is $\mathrm{S_p}$-closed space if possible suppose that X is not $\mathrm{S_p}$-closed space, then by Proposition 3.8 there exists an $\mathrm{S_p}$-regular cover of X say $\{V_\alpha : \alpha \in \Delta\}$ which has no finite subcover, this implies that for every finite subset Δ_0 of Δ, $X \neq \bigcup_{\alpha\in\Delta_0} V_\alpha$ and then $\bigcap_{\alpha\in\Delta_0} V_\alpha^c \neq \phi$. But for each $\alpha \in \Delta$, $V_\alpha^c \in \mathrm{S_pR}(X)$, then $V_\alpha^c \in \mathrm{S_pO}(X)$ implies that by Lemma 3.25 V_α^c is $\mathrm{S_p}$-θ-closed set in X for each $\alpha \in \Delta$. Then the family $\{V_\alpha^c : \alpha \in \Delta\}$ is $\mathrm{S_p}$-θ-closed sets of X with finite intersection property, so by hypothesis $\bigcap_{\alpha\in\Delta} V_\alpha^c \neq \phi$ implies that $X \neq \bigcup_{\alpha\in\Delta} V_\alpha$ which is a contradiction, so X must be $\mathrm{S_p}$-closed space.

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