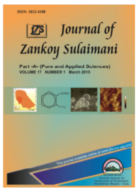
Journal homepage www. jzs.uos.edu.krd
Journal of Zankoi Sulaimani
Part-A- (Pure and Applied Sciences)



JCGP-Injective Rings With Some Types of Rings

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Article info

Original: 13 Sep. 2014
Revised: 13 Nov. 2014
Accepted: 10 Dec. 2014
Published online:
20 March 2015

Key Words:
JCP-injective rings
JCGP-injective rings
GP-injective rings
regular rings

Abstract

The concept of right P-injective rings, which was first introduced by Ming in 1974. As a generalization of right P-injective rings, the notion of right JCP-injective rings, which was first introduced by Junchao in 2009. Now, in the present paper we introduce the notion of right JCGP-injective rings, which is stronger than right GP-injective rings due to Ming in 1985. Some properties, characterizations and main results for right JCGP-injective rings will be finding.

I. Introduction and Preliminaries

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary. We write M_R and $_RM$ to indicate a right and left R-module,respectively. The Jacobson radical [1] of a ring R, denoted by J(R) is the intersection of all maximal ideals of R. For a subset X of R, the right annihilator of X in a ring R is defined by $r(X) = \{t\in R: xt = 0, \text{ for all } x\in X\}$. Similarly, define the left annihilator of X in a ring R as $\ell(X)=\{ t\in R: tx = 0, \text{ for all } x\in X\}$. If $X = \{a\}$, we usually us to the abbreviation $r(a)(\ell(a))$. An ideal I of a ring R is said to be essential if and only if I has a non-zero intersection with every non-zero ideal of R. Let R be a ring and x be an element in R. Then x is said to be a right singular if and only if r(x) is an essential right ideal of R. The set of all right sigular elements in R is denoted by Y(R). Y(R) is a right ideal in R, which is called the right singular ideal of R. A left singular ideal Z(R) is similarly defined. A ring R is called right (left) non-singular if Y(R) = (0) (Z(R) = (0)). A ring R is called reduced [2] if it contains no non-zero nilpotent elements, or equivalently, $a^2 = 0$ implies $a = 0$, for all $a \in R$. Two idempotents a and b are called orthogonal [2] if $ab = ba = 0$. A ring R is semi-simple [2] if it is semi-simple as a (left) module over itself, or equivalently, if every (left) module over it is semi-simple (every module is a sum of its cyclic submodules). A ring R is semi-local [7] if R / J(R) is a semi-simple ring. A ring with identity is called semi-primary [7] if R / J(R) is a semi-simple and J(R) is a nilpotent ideal. A module M has finite Goldie dimension [14] if M does not contain a direct sum of infinite number of non-zero submodules. A right Noetherian (Artinian) ring [2] is a ring that satisfies the ascending (descending) chain condition on right ideals. $N \mid M$ will mean that submodule N is a direct summand of M. An element a of a ring R is said to be regular [16] if there exists an element $b\in R$ such that $a = aba$. A ring R is called von Neumann regular (briefly, regular) [16] if every element of R is regular. A ring R is called π-regular [8] if for every element $a\in R$, there exists a positive integer n and $b\in R$ such that $a^n = a^n\, b\, a^n$. A ring R is called right SPP [12] if for any $a\notin Y(R)$, aR is projective. A right R-module M is called right principally injective (briefly , right P-injective) [9] if for any principal right ideal aR of R and any right R-homomorphism of aR into M extends to one of R into M. R is called right P-injective if the right R-module R_R is P-injective. A right R-module M is called Generalized P-injective (briefly, GP-injective) [10] if for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^nR into M extends to one of R into M. Recall that a ring R is called right (left) GP-injective if the right (left) R-module R_R ($_RR$) is GP-injective. A right R-module M is called JCP-injective [4] if for each $a \in R \setminus Y(R)$, every right R-homomorphism from aR to M can be extended to one of R into M. If R_R is JCP-injective, then R is a right JCP-injective ring [4]. Or, equivalently, a ring R is called right JCP-injective [4] if for any right non-singular element c of R and any right R-homomorphism $g : cR \to R$, there exists $m \in R$ such that $g(ca) = mca$, for all $a \in R$.

II. Right JCGP-Injective Rings

In this section we introduce a new generalization of JCP-injective rings, which is due to Junchao W. and it is called JCGP-injective rings and we obtain some characterizations and properties of these types of rings.

We start this section with the following definition.

Definition 2.1:

A right R-module M is called JCGP-injective if for each $a \in R \setminus Y(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to M can be extended to one of R into M. Equivalently, a ring R is called right JCGP-injective if for any right non-singular element c of R such that $c^n \neq 0$, for some positive integer n and any right R-homomorphism $g: c^nR \to R$, there exists $m \in R$ such that $g(c^na) = mc^na$, for all $a \in R$.

If R_R is JCGP-injective, we call R as a right JCGP-injective ring.

The following result is a special chracterization of right JCGP-injective rings.

Theorem 2.2:

The following statements are equivalent for a ring R.

(i) R is right JCGP-injective.

(ii) $\ell r(a^n) = Ra^n$, for each $a \notin Y(R)$ and a positive integer n.

(iii) $r(a^n) \subseteq r(b^n)$, for $a, b \in R$, $a \notin Y(R)$ and a positive integer n implies that $Rb^n \subseteq Ra^n$.

(iv) $\ell(b^nR \cap r(a^n)) = \ell(b^n) + Ra^n$, for $a, b \in R$ with $ab \notin Y(R)$ and a positive integer n.

Proof:

(i) $\Longrightarrow$ (ii). Clearly, $Ra^n \subseteq \ell r(a^n)$, for all $a \notin Y(R)$ and a positive integer n.

Now, let $x \in \ell r(a^n)$ and define $f: a^nR \to R$ by $f(a^nt) = xt$, for some positive integer n. Then, f is a well-defined right R-homomorphism because $r(a^n) \subseteq r(x)$. By (i), for some $c \in R$ and a positive integer n, $x = f(a^n) = ca^n \in Ra^n$, which implies that $\ell r(a^n) \subseteq R(a^n)$ and hence $\ell r(a^n) = Ra^n$.

(ii) $\Longrightarrow$ (iii). If $r(a^n) \subseteq r(b^n)$ with $a, b \in R$, $a \notin Y(R)$ and a positive integer n, then $Rb^n \subseteq \ell r(b^n) \subseteq \ell r(a^n)$. Since $a \notin Y(R)$, by (ii), $\ell r(a^n) = Ra^n$. Hence, $Rb^n \subseteq Ra^n$.

(iii) $\Longrightarrow$ (iv). Clearly $\ell(b^n) + Ra^n \subseteq \ell(b^nR \cap r(a^n))$, for some positive integer n.

Now, let $x \in \ell(b^nR \cap r(a^n))$. Then, $r(a^n b^n) \subseteq r(x b^n)$. Since $ab \notin Y(R)$, $Rx b^n \subseteq R a^n b^n$ by (iii). So, $x b^n = c a^n b^n$, for some $c \in R$ and a positive integer n. Hence, $x - ca^n \in \ell(b^n)$, as required.

(iv) $\Longrightarrow$ (i). Let $a \notin Y(R)$ and $f: a^nR \to R$ be any right R-homomorphism, for some positive integer n.

Clearly, $r(a^n) \subseteq r(f(a^n))$. So, $f(a^n) \in \ell r(f(a^n)) \subseteq \ell r(a^n) = \ell(1R \cap r(a^n)) = \ell(1) + Ra^n = Ra^n$ by (iv) because $a1 \notin Y(R)$, that is , $f(a^n) \in Ra^n$. Write $f(a^n) = ca^n$, for some $c \in R$ and a positive integer n. Then, we can define $g: R_R \to R_R$ by $g(t^n) = c\, t^n$, $t \in R$ and a positive integer n, obviously, $g|_{aR} = f$. Consequently, R is a right JCGP-injective. ♦

Corollary 2.3:

If R is a right non-singular , then R is right GP-injective if and only if R is right JCGP-injective.

Recall that a ring R is called right Generalized P.P. [3] if for any $a \in R$, a^nR is projective, for some positive integer n.

Corollary 2.4:

A ring R is π-regular if and only if R is right Generalized P.P. ring and right JCGP-injective ring.

Lemma 2.5:

Every right GP-injective ring is right JCGP-injective. The converse of the above lemma is not true in general as it is shown in the example.

Example 2.6:

Let V be a two-dimensional vector space over a field F. The trivial extension $R = T(F,V) = F \oplus V$ is commutative, local Artinian ring with $J^2(R) = (0)$ and $J(R) = Y(R)$ [9]. But R is not P-injective ring [13] and hence it is not a GP-injective ring.

On the other hand, if $x \in R$ with $x \notin Y(R)$, then x is invertible. So, $\ell r(x^n) = R = Rx^n$, for some positive integer n. This implies that R is a right JCGP-injective. Hence there exists a right JCGP-injective ring, which is not right GP-injective.

Theorem 2.7:

Let R be a right JCGP-injective. Let $c \in R$ such that $r(c^n) = (0)$, for some positive integer n. Then c is left invertible.

Proof:

Let $c \in R$ such that $r(c^n) = (0)$, for some positive integer n. Then, $c \notin Y(R)$ and so, by Theorem 2.2 (ii), $R = \ell r(c^n) = Rc^n$. Hence, $1 = sc^n = sc^{n-1}c$. Set, $t = sc^{n-1}$. Therefore, $1 = tc$. So, c is left invertible.♦

Theorem 2.8:

Let R be a right JCGP-injective. Then, $Y(R) \subseteq J(R)$.

Proof:

Let $z \in Y(R)$ and $a \in R$. Then, $r((1-az)^n) = (0)$, for some positive integer n. Therefore, by Theorem 2.7, 1-az is left invertible. So, $v(1-az) = 1$, for some $v \in R$. This proves that $z \in J(R)$. ♦

Theorem 2.9:

Let R be a right JCGP-injective. If $c \in R$ such that $dc^n = 1$, for some $d \in R$ and a positive integer n, then $M = Mc = cM$, for any right or left R-module M.

Proof:

If $c \in R$ such that $dc^n = 1$, for some $d \in R$ and a positive integer n. Now, $\ell(c^n) = (0)$, then by Theorem 2.7, c is left invertible. Thus , $M = Mdc \subseteq Mc \subseteq M$ implies that $M = Mc$, for any right R-module M. Likewise, $M = cM$, for any left R-module M. ♦

Theorem 2.10:

Let R be a right JCGP-injective. If P is a reduced principal right ideal of R, then $P = eR$, where $e = e^2 \in R$ and $(1-e)R$ is an ideal of R.

Proof:

Let P be a non-zero reduced principal right ideal c^nR, for some $c \in R$ and a positive integer n. Since $c^n \in Y(R)$, $\ell r(c^n) = Rc^n$. Hence $r(c) = r(c^n)$, shows that $Rc = \ell r(c) = \ell r(c^n) = Rc^n$. Therefore, $c = bc^n$, for some $b \in R$, which implies that $c = c^{n-1} b c$. Because P is reduced, hence P is generated by the idempotent element $e = c^{n-1} b$. Also, for any $a \in R$, $(ea-eae)^2 = 0$ implies $ea = eae$, whence $eR(1-e) = (0)$. Therefore, $R(1-e) \subseteq (1-e)R$. ♦

In order to prove the next result, we need the proof of the following for GP-injective rings.

Theorem 2.11:

Let R be a right GP-injective with $a, b \in R$ and a positive integer n. If $a^nR \cong b^nR$ and $b^nR \mid R$ then $a^n R \mid R$.

Proof:

Suppose $b^n R = eR$, where $e^2 = e$, and $\sigma: a^nR \rightarrow b^nR$ is an isomorphism. Let $\sigma(a^n) = b^nd$ and $\sigma^{-1}(e) = a^nc$, for some positive integer n and c, $d \in R$. Then, $b^ndc = \sigma(a^nc) = e$, so $f = c\, b^nd$ is an idempotent and $a^n f = \sigma^{-1}(b^nd) = a^n$, whence $Ra^n \subseteq Rf$. This equality because $r(a^n) \subseteq r(f)$ (Indeed, $f(t) = c\sigma(a^nt)$, for all $t \in R$ and a positive integer n). ♦

Recall that a ring R is right C_2 [13] if every right ideal T, which is isomorphic to a direct summand of R_R is a direct summand.

In [13], it is shown that right P-injective rings are right C_2.

Theorem 2.12:

Let R be a right JCGP-injective. Then R is right C_2.

Proof:

Assume that $a^nR \cong eR$, where a, $e^2 = e \in R$ and a positive integer n. Then, $a \notin Y(R)$, so $Ra^n = \ell r(a^n)$ by Theorem 2.2 (ii). Since $a^nR \cong eR$, then there exists $t^2 = t \in R$ such that $a^nt = a^n$ and $r(a^n) = r(t)$. Hence, $Ra^n = \ell r(a^n) = Rt$. Consequently, by Theorem 2.11, $a^nR \mid R$. This means that a^nR is a direct summand of R. ♦

Theorem 2.13:

Let R be a right JCGP-injective. If $a^nR \mid R$, $b^nR \mid R$ with $a^nR \cap b^nR = (0)$, for all $a, b \in R$ and a positive integer n, then $(a^nR \oplus b^nR) \mid R$.

Proof:

Follows from Theorem 2.12. ♦

Theorem 2.14:

Let R be a right JCGP-injective. Then the following statements are equivalent for $a \notin Y(R)$.

(i) a^nR is projective.

(ii) $a^nR \mid R$.

(iii) a^nR is a JCGP-injective module.

Proof:

By Theorem 2.12, we have (i) $\Leftrightarrow$ (ii). Obviously, (iii) $\Rightarrow$ (ii) always hold. (ii) $\Rightarrow$ (iii), we only need to show that $\ell_{a^nR}\, r_R(b) = a^nRb$, for all $b \notin Y(R)$ and a positive integer n. In fact, if $a^nc \in \ell_{a^nR}\, r_R(b)$, then $r_R(b)$

$\subseteq r_R(a^n c)$, so $a^n c \in \ell_R\, r_R(a^n c) \subseteq \ell_R r_R(b) = Rb$. Because $a^n R = eR$, for some $e^2 = e \in R$, $a^n c = ea^n c = eRb$. Hence $\ell_{a^n R}\, r_R(b) \subseteq a^n Rb$, which shows that $\ell_{a^n R}\, r_R(b) = a^n Rb$. ♦

III. Main Results:

In this section we verify some main results for JCGP-injective rings. Also, compare it with some types of rings such as directly finite rings, semi-local rings, abelian rings, Noetherian rings and Artinian rings.

Recall that a module M_R is GC_2 [13] if $N \subseteq M$ with $N_R \cong M$ implies that $N \mid M$.

A ring R is right GC_2 if R_R is GC_2. Clearly, every right C_2 ring is right GC_2[4].

Yiqiang Zhou, shows that if M_R is GC_2 and finite dimensional, then End (M_R) is a semi-local ring.

Theorem 3.1:

Let R be a right JCGP-injective ring.

1. If R_R is of finite Goldie dimensional, then R is a semi-local ring.
2. If J(R) is nilpotent, then R is right Noetherian if and only if R is right Artinian.

Proof:

(1) Since R is right JCGP-injective ring , then by Theorem 2.12, R is right C_2 and hence R is right GC_2. Hence, $R \cong End(R_R)$ is semi-local because R_R is finite Goldie dimensional.

(2) If R is right Noetherian, then by (1), R is a semi-local ring. Hence R is semi-primary because J(R) is nilpotent. Whence, R is right Artinian. ♦

Recall that a ring R is directly finite [15] if uv = 1 in R implies that vu = 1. For example, semi local rings are directly finite [4].

Obviously, R is directly finite if and only if every epimorphism $R_R \rightarrow R_R$ is an isomorphism [4]. It is known that [4]

1. If each monomorphism $R_R \rightarrow R_R$ is an isomorphism, then R is a directly finite;
2. R is directly finite if and only if R/J(R) is directly finite. ♦

Theorem 3.2:

Let R be a right JCGP-injective ring. Then the following statements are equivalent:

(i) R / J(R) is directly finite.

(ii) Every monomorphism $R_R \rightarrow R_R$ is an isomorphism.

(iii) R is directly finite.

If every complement right ideal of R is not singular, then the conditions above are also equivalent to.

(iv) R / Y(R) is directly finite.

Proof:

(ii) ⇒ (iii). It is obvious.

(iii) ⇔ (i) and (iv) ⇒ (iii) are obvious.

(iii) ⇒ (ii). Assume that $f: R_R \rightarrow R_R$ is a monomorphism. Since R is right GC_2, Im $(f) = eR$, for some $e^2 = e$ because $Im(f) \cong R_R$. Write $f(1) = a^n$, for some $a \in R$ and a positive integer n, then $a^n R = f(R) = eR$. Hence, $a^n = ea^n = a^n ba^n$, for some $b \in R$ and a positive integer n with $e = a^n b$. Thus, $ba^n = 1$ because $r(a^n) = (0)$. By (iii), $a^n b = 1$, so $f(R) = a^n R = eR = a^n bR = R$. This implies that f is an epimorphism.

(iii) ⇒ (iv). Let $a, b \in R$ such that $1-ab \in Y(R)$. Since R is right JCGP-injective, then by Theorem 2.8, $(1-ab) \in J(R)$. Let $ab = 1+x$, for some $x \in J(R)$. So, $ab(1+x)^{-1} = 1$. Since R is directly finite, then $b(1+x)^{-1}a=1$. If $x \notin Y(R)$, then there exists a non-zero right ideal I of R, which is maximal with respect to the property that $I \cap r(x) = (0)$. By hypothesis, there exists $b \in I$ such that $b \notin Y(R)$. Hence $xb \notin Y(R)$. Let $f: xb^n R \rightarrow R$ be defined by $f(xb^n t) = b^n t$, for all $t \in R$ and a positive integer n. Then f is a well defined right R-homomorphism.

Since R is right JCGP-injective, $f = c$, $c \in R$. Hence $b^n = f(xb^n) = cxb^n$ and so $(1-cx)b^n = 0$. Since : $cx \in J(R)$, $b^n = 0$, which is a contradiction. Hence $x \in Y(R)$, which implies that $1- b^n a \in Y(R)$, so $R / Y(R)$ is directly finite. ♦

In fact, from the proof of Theorem 3.2, we know that every monomorphism $R_R \to R_R$ is an isomorphism if and only if R is directly finite and right GC_2. ♦

Recall that a ring R is abelian [6] if every idempotent element of R is central. Obviously, every abelian ring is directly finite.

Recall that a ring R is called I-finite [15] if it contains no infinite set of orthogonal idempotent. Obviously, every I-finite rings are also directly finite.

In [5], it is proved that if R is right GP-injective, then J(R) =Y(R). We do not know whether the result holds for right JCGP-injective. But, from the proof of Theorem 3.2, we can obtain the following corollary.

Corollary 3.3:

Let R be a right JCGP-injective ring. Then:

(1) If R satisfies one of the following conditions, then every monomorphism $R_R \to R_R$ is an isomorphism.

(i) R is abelian

(ii) R is I-finite

(iii) R is semi-local.

(2) If each non-zero complement right ideal of R is not contained in Y(R), then

(i) J(R) =Y(R).

(ii) for each $a \in R \setminus J(R)$, there exists $c \in R$ such that the inclusion $r(a^n) \subseteq r(a^n - a^n c a^n)$ is proper, for some positive integer n.

Definition 3.4:

A ring R is called right Generalized SPP (briefly, GSPP) if for any $a \notin Y(R)$, there exists a positive integer n such that $a^n R$ is projective.

Theorem 3.5:

A ring is right GSPP if and only if every homomorphic image of a right JCGP-injective R-module is JCGP-injective.

Proof:

The proof is likewise to [17, Theorem 2.10 (1)]. ♦

Theorem 3.6:

Let R be a right JCGP-injective and right GSPP ring. Then:

(i) $Z(R) \subseteq J(R) = Y(R)$.

(ii) For each $a \notin Y(R)$, $a^n = a^n b a^n$, for some $b \in R$ and a positive integer n, then R / J(R) is π-regular.

Proof:

(i) First we show that $J(R) \subseteq Y(R)$. Suppose this is not the case. Then, there exists $a \in J(R)$ such that $b \notin Y(R)$. So, $r(a^n) = r(e)$, for some, $e^2 = e \in R$ and a positive integer n because R is a right GSPP ring. Since R is right JCGP-injective, then $Re = \ell r(e) = \ell r(a^n) = Ra^n \subseteq J(R)$. This is a contradiction. Likewise, we can show that $Z(R) \subseteq Y(R)$. By Theorem 2.8, we have $Z(R) \subseteq J(R) = Y(R)$.

(ii) Let $a \notin Y(R)$. Then, $Ra^n = Re$, so $a^n = a^n e \in a^n R a^n$. Hence, R / J(R) is π-regular. ♦

Theorem 3.7:

The following statements are equivalent for a ring R.

(i) R satisfies the condition, $a^n \in a^n R a^n$, for all $a \notin Y(R)$ and any positive integer n.

(ii) R is right JCGP-injective and right GSPP ring.

(iii) Every right R-module is JCGP-injective.

(iv) Every cyclic right R-module is JCGP-injective.

Proof:

(ii) $\Rightarrow$ (i). Follows from Theorem 3.6 (ii).

(iii) $\Rightarrow$ (iv). It is obvious.

(i) $\Rightarrow$ (iii). Let M be any right R-module, $a \in R$ with $a \notin Y(R)$, there exists a positive integer n such that $a^n \neq 0$ and f: $a^nR \to M$ any right R-homomorphism. By (i), $a^n = a^nba^n$, for some $b \in R$ and a positive integer n. Let $a^n\, b = e$ and $f(e) = m$, where $m \in R$. Then h: $R \to M$ defined by $g(t) = mt$, $t \in R$ is a right R-homomorphism and $g(a^nt) = m\, a^n\, t = f(e)\, a^n\, t = f(a^nb)\, a^nt = f(a^n\, b\, a^n)\, t = f(a^n)t = f(a^n\, t)$, so M is JCGP-injective.

(iv) $\Rightarrow$ (i). Let $a \notin Y(R)$. By (iv), a^nR is JCGP-injective, for some positive integer n, so the identity map $a^nR \to a^nR$ can be extended to one of R into R. Hence, $a^n = a^nba^n$, for some $b \in R$ and a positive integer n. ♦

Theorem 3.8:

Let e be an idempotent of R such that ReR = R and let S = eRe. If R is right JCGP-injective, then so is S.

Proof:

Let $x \in S \setminus Y(S)$. Then $x \notin Y(R)$. Otherwise, there exists an essential right ideal I of R such that $xI = (0)$. Since $eR \cap I \neq (0)$, so we have $eI \neq (0)$. Since R = ReR, so we have $eI = eIR = eIReR = eIeR$, $eIe \neq (0)$. We claim that eIe is an essential right ideal of S. Let K be any non-zero right ideal of S. Then $KR \cap I \neq (0)$. Let $0 \neq y \in KR \cap I$ and let $1 = \sum_{i=1}^{n} a_i\, e\, b_i$, $a_i, b_i \in R$. Then, $y = y\,1 = \sum_{i=1}^{n} y\, a_i\, e\, b_i$, so there exists some $i_0 \in \{1, 2, 3, \ldots, n\}$ such that $y\, a_{i_0}\, e \neq 0$. Since $e\, y\, a_{i_0}\, e = y\, a_{i_0}\, e \in K\, R\, a_{i_0}\, e \cap I \subseteq K \cap I = K \cap eIe$, eIe is an essential right ideal of S. Since $xeIe = xIe = (0)$, so $x \in Y(S)$, which is a contradiction. Hence, $x \notin Y(R)$. Since R is right JCGP-Injective, so $\ell_R\, r_R(x^n) = Rx^n$, for some positive integer n. Now, let $z \in \ell_S\, r_S(x^n)$. Then, $r_S(x^n) \subseteq r_S(z)$. Let $a \in r_R(x^n)$. Then, $x^na = 0$, so $0 = x^na\, a_ie = x^ne\, a\, a_i\, e$, $i = 1, 2, \ldots, n$. Hence, $eaa_ie \in r_S(x^n) \subseteq r_S(z)$, so $za\, a_ie = zea\, a_ie = 0$. Thus, $za = \sum_{i=1}^{n} z\, a\, a_i\, e\, b_i = 0$, so $a \in r_R(z)$. This shows that $r_R(x^n) \subseteq r_R(z)$. So, $z \in \ell_R r_R(z) \subseteq \ell_R\, r_R(x^n) = Rx^n$. Hence, $z = eze \in eRex^n = Sx^n$, which implies that $\ell_S\, r_S\,(x^n) \subseteq Sx^n$ and so $\ell_S\, r_S\,(x^n) = Sx^n$. Hence, S is right JCGP-injective. ♦

Theorem 3.9:

Let R be a right JCGP-injective ring and let $a, b \in R$ with $b \notin Y(R)$. Then:

(1) If b^nR embeds in a^nR, then Rb^n is a quotient of Ra^n.

(2) If a^nR is a quotient of b^nR, then Ra^n embeds in Rb^n.

(3) If $b^nR \cong a^nR$, then $Ra^n \cong Rb^n$,

(4) If I, J are right ideals of R with $I \cap (J + Y(R)) = (0)$ and I is an ideal of R, then $\mathrm{Hom}_R(I_R, J_R) = (0)$.

Proof:

(1) Let σ: $b^nR \to a^nR$ be a monomorphism, for some positive integer n. Since R is right JCGP-injective and $b \notin Y(R)$ we can let $\sigma = v$, $v \in R$. Then $vb^n = a^nu$, $u \in R$ and a positive integer n, so define ψ: $Ra^n \to Rb^n$ by $\psi(ta^n) = ta^nu = tvb^n$, then ψ is a well-defined left R-homomorphism. Since $vb^n \notin Y(R)$ and $r(vb^n) = r(b^n)$, $Rb^n = \ell\, r(b^n) = \ell\, r(vb^n) = Rvb^n$. Hence, clearly ψ is epimorphism.

(2) Let σ: $b^nR \to a^nR$ be an epimorphism and let v, u and ψ be as in (1). Write $a = \sigma(b^ns) = vb^ns$, $s \in R$, for some positive integer n. Then, $\psi(ta^n) = 0$ gives $0 = ta^nu = tvb^n$, whence $ta^n = tvb^ns = 0$. Hence ψ is monomorphism.

(3) Follows from the proof of (1) and (2).

(4) If there exists $0 \neq f \in \mathrm{Hom}_R(I_R, J_R)$, then there exists $0 \neq a \in I$ such that $f(a^n) \neq 0$, for some positive integer n. Then $f(a^n) = va^n$, where $v \in R$ and a positive integer n because $I \cap Y(R) = (0)$. Since I is an ideal, $va^n \in I$. Hence $f(a^n) \in I \cap J = (0)$, which is a contradiction. Hence $\mathrm{Hom}_R(I_R, J_R) = (0)$. ♦

Recall that a ring R is called right Kasch if every simple right module can be embedded in R, or equivalently, if $\ell(M) \neq (0)$, for every maximal right ideal M of R, or equivalently, a ring R is called right Kasch ring [11] if every maximal right ideal of R is a right annihilator.

Theorem 3.10:

Let R be a right JCGP-injective ring. If R is a right Kasch ring, then $Rb^n \cap \ell(J) \neq (0)$, for $0 \neq b \in R$, $b \notin Y(R)$ and a positive integer n.

Proof:

Let $0 \neq b \in R$, $b \notin Y(R)$ and choose M maximal in b^nR, for some positive integer n. Let $\sigma: b^nR / M \to R_R$ is a monomorphism. If $\gamma: b^nR \to R$ is defined by $\gamma(x^n) = \sigma(x^n + M)$, then $\gamma = a^n$, where $a \in R$ because $b \notin Y(R)$. So, $a^nb^n = \gamma(b^n) = \sigma(b^n + M) \neq (0)$. But $a^nb^nJ = \sigma(b^nJ) = (0)$ because $(b^nR / M)J = (0)$. Therefore, $(0) \neq a^nb^n \in Rb^n \cap \ell(J)$. This completes the proof. ♦

Finally, we give the following result about right JCGP-injective ring.

Theorem 3.11:

Let R be a right JCGP-injective ring. If R_R is an essential in X_R, where $X_R \cong R_R$, then $X = R$.

Proof:

Let $f: R_R \to R_R$ be the isomorphism and $f(1) = b^n \in X$, for some positive integer n. Then, $b^nR = \mathrm{Im}(f) = X$. Since $1 \in R \subseteq X$, let $1 = b^nu$, $u \in R$. Hence, $R_R = 1R = b^nuR$ and $r(u) = (0)$. Since R is right JCGP-injective, then by Theorem 2.7, there exists $d \in R$ such that $du = 1$. Let $e = ud$. Then, $e^2 = e$ and $uR = eR$. Hence, $R = b^nuR = b^neR$. It is clear that $X = b^nR = b^n(eR \oplus (1-e)R) = b^neR + b^n(1-e)R$.

If $x \in b^neR \cap b^n(1-e)R$, then there exist $t_1, t_2 \in R$ such that $x = b^net_1 = b^n(1-e)t_2$, so $f^{-1}(x) = et_1 = (1-e)t_2$. Hence, $f^{-1}(x) = 0$ and then $x = 0$, so $X = b^nR = b^neR \oplus b^n(1-e)R = R \oplus b^n(1-e)R$. Since R_R is essential in X_R, $b^n(1-e)R = (0)$ and so $X = b^neR = R$. ♦

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