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Nonstandard Linear Combination of the Cantor Sets

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ABSTRACT

This paper investigated the linear combinations of the Cantor set within the framework of nonstandard analysis for studying linear combinations of Cantor sets and apply it to the Minkowski sum $C^* + mC^*$, where $m \in (0,1)$ and C^* denotes the Cantor set at an unlimited stage of its construction. The primary aim is to characterize the structure of such sums at infinitesimal resolution and to relate them rigorously to their classical counterparts. Our main result shows that $C^* + mC^*$ is infinitely close to the interval $[0,1+m]$ in the sense of standard parts, while internally retaining a highly fragmented structure composed of infinitesimal intervals. This provides a precise nonstandard counterpart to classical results on sums of Cantor sets. The novelty of the paper lies in exploiting infinitesimal methods to capture both the microstructure and macroscopic behavior simultaneously, without resorting to limiting arguments. This yields a unified perspective on fractal sums, clarifies the role of infinitesimal gaps, and demonstrates the effectiveness of nonstandard analysis in resolving fine-scale geometric properties. The approach opens new avenues for the study of fractal sets and their applications in analysis and related fields.

1. Introduction

The Cantor set, introduced by Georg Cantor in 1883, occupies a central place in analysis, topology, fractal geometry, and measure theory. It is traditionally obtained by repeatedly deleting the open middle third from subintervals of the unit interval $[0,1]$. Although its construction is elementary, the Cantor set exhibits a range of remarkable properties: it is uncountable, perfect, nowhere dense, and nevertheless has Lebesgue measure zero. These characteristics make it a central example in the study of fractal structures and singular measures, as discussed in detail by (Massopust, 2017) and further surveyed from a topological perspective by (Patiño Ortiz, 2023; Soltanifar, 2025). Consequently, the Cantor set provides a natural framework for investigating the interplay between geometric structure, measure theory, and dimensional analysis. A major direction of research concerns the behavior of linear combinations and Minkowski sums of Cantor sets. Classical results, $C \oplus C = [0,2]$, show that such combinations may either produce intervals or retain a fractal structure, depending on the parameters involved. (Pawłowicz, 2013) provided a comprehensive analysis and systematic treatment of linear combinations $aC \oplus bC$ for real coefficients a, b, reducing the general case to the study of $C \oplus mC$ for $m \in (0,1)$. Earlier and subsequent works, including (Cabrélli et al., 1997; Anisca and Ilie, 2001; Takahashi, 2019; Wang et al.,

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2021), further explored structural, dynamical, and measure-theoretic aspects of these sums.

These studies highlight the delicate balance between overlap and gap formation, which determines whether the resulting set fills an interval or preserves fractal features. In a different direction, nonstandard analysis, introduced by (Robinson, 1996) and formalized via Internal Set Theory by (Nelson, 1977), provides a rigorous framework for working with infinitesimal and unlimited quantities. Foundational expositions by (Diener and Diener, 1995; Cutland 1997), and (Goldbring, 2014) demonstrate how this approach allows one to analyze limiting processes and fine-scale structures without relying on classical ε–δ techniques. In particular, nonstandard methods have proven effective in studying compactness, convergence, and local behavior in a unified manner. Recent developments have extended the Cantor set to the nonstandard setting. In this framework, the Cantor set at an unlimited stage of construction consists of an internal union of 2^{ω} intervals of infinitesimal length $3^{-\omega}$, providing a refined description of its microstructure. Such an approach has been explored, for example, by (Jumha et al., 2024), who showed that the nonstandard Cantor set possesses a measure that is infinitely close to zero while retaining a rich internal structure. This perspective opens new possibilities for analyzing fractal sets at infinitesimal resolution.

The present work is motivated by the problem of understanding how linear combinations of Cantor sets behave in the nonstandard framework, and how their infinitesimal structure relates to their global properties. While classical results describe the limiting behavior of such sums, the nonstandard approach enables us to study these objects directly at the level of infinitesimal scales. This leads to a more precise understanding of how local fragmentation interacts with global interval formation. The main objective of this paper is to investigate the Minkowski sum $C^* + mC^*$, where C^* denotes the nonstandard Cantor set and $m \in (0,1)^*$. Our principal result shows that this sum is infinitely close to the interval $[0,1+m]$ in the sense of standard parts, while internally remaining a union of infinitesimal intervals. This extends the classical results of (Pawłowicz, 2013) to the nonstandard setting and provides a rigorous formulation of the relationship between fractal structure and interval behavior. The novelty of this study lies in the systematic application of nonstandard analysis to linear combinations of fractal sets, allowing simultaneous access to both infinitesimal and global properties. Unlike classical approaches, which rely on limiting arguments, our method operates directly within the hyperreal framework, simplifying the analysis and revealing structural features that are not immediately visible in the standard setting.

Furthermore, the results obtained here offer important interpretations. They demonstrate that fractal sets can exhibit a form of "hidden continuity," where infinitesimal gaps persist internally while the global structure approximates a full interval. This dual behavior provides new insight into the geometry of Cantor-type sets and illustrates the effectiveness of nonstandard methods in capturing fine-scale phenomena. Such insights are consistent with broader applications of multiscale analysis in complex systems, as discussed in recent applied studies (e.g., Chen and Lv, 2022; Bordonhos et al., 2023; Capone et al., 2024).

- Extending this analysis to the nonstandard setting allows us to:
- Rigorously establish the structure of Minkowski sums of nonstandard Cantor sets.
- Characterize the distribution of points in such combinations at infinitesimal scales.
- Expose the preservation of fractal structure under nonstandard operations.

The paper is organized as follows. Section 2 reviews the necessary preliminaries from nonstandard analysis and Cantor set theory. Section 3 presents the main results on linear combinations of nonstandard Cantor sets, including detailed proofs using transfer principles and interval decomposition techniques. Finally, section 4 discusses further implications and potential applications of the results.

2. Preliminaries

2.1. Nonstandard Analysis Foundations

We use the framework of Internal Set Theory (IST) as developed by (Nelson, 1977), which extended the standard Zermelo-Fraenkel set theory with the addition of the predicate $st(\cdot)$ denoting "standard" and three additional Principles: Transfer (T), Idealization (I), and Standardization (S).

A mathematical expression is internal if it does not involve nonstandard concepts (such as standard, infinitesimal, limited, and unlimited). Otherwise, it is external with the following correspondence principles. Nelson's framework for nonstandard analysis rests on three fundamental principles:

Transfer Principle (T): For any internal formula A with free variables $\mathrm{x}, \mathrm{t}_1, \mathrm{t}_2, \ldots, \mathrm{t}_\mathrm{n}$:

$$\forall^{\mathrm{st}}\mathrm{t}_1 \ldots \forall^{\mathrm{st}}\mathrm{t}_\mathrm{n}\, [\forall^{\mathrm{st}}\mathrm{x}\, \mathrm{A} \Leftrightarrow \forall \mathrm{x}\, \mathrm{A}].$$

Idealization Principle (I): For any internal formula A:

$$\forall^{\mathrm{stfin}}\mathrm{x}'\, \exists \mathrm{y}\, \forall \mathrm{x} \in \mathrm{x}'\ \ \mathrm{A} \Leftrightarrow \exists \mathrm{y}\, \forall^{\mathrm{st}}\mathrm{x}\, \mathrm{A}.$$

Standardization Principle (S): For any formula F with free variables z and possibly others:

$$\forall^{st}X\, \exists^{st}Y\, \forall^{st}z[z \in Y \Leftrightarrow z \in X \wedge F(z)].$$

Where; $\forall^{st}$, $\exists^{st}$, and $\forall^{stfin}$ are refers to; for all standard, there exit standard, and for all standard finites set respectively.

Definition 2.1. (Robinson, 1996; Hamad and Ismail, 2011)

In the hyperreal number system R^*, element $x \in \mathbb{R}^*$ is

i. Infinitesimal if $|x| < r$ for all positive real $r \in R^+$,

ii. Limited if $|x| < r$ for some real $r \in R^+$,

iii. Unlimited if $|x| > r$ for all real $r \in R^+$.

Definition 2.2. (Goldbring, 2014)

For $x, y \in \mathbb{R}^*$, we say x and y are infinitely close, denoted by $x \simeq y$, if $x - y$ is infinitesimal.

Lemma 2.3. (Cutland 1997)

In the hyperreal number system $\mathbb{R}^*$:

i. The sum and product of infinitesimals are infinitesimal.

ii. The product of an infinitesimal and a limited number is infinitesimal.

iii. The reciprocal of a nonzero infinitesimal is unlimited.

iv. Zero is the only standard infinitesimal.

In this paper the direct sum of two Cantor sets is described with nonstandard coefficient $m \in (0,1)^*$.

Theorem 2.4. (Existence of Standard Parts) (Diener and Diener, 1995)

If r is limited hyperreal number, then there exists a unique $\mathrm{s} \in \mathrm{R}^*$ such that $r \simeq s$. The number s is called the standard part or (shadow) of r and denoted by $st(r)$.

Definition 2.5. (Kuttler and Farah, 2017)

Let $A, B \subseteq \mathbb{R}$ be two sets and m be a constant, we define the two operations $\oplus$ and "." as follows:

i. $A \oplus B = \{x + y, x \in A, y \in B\}$.

ii. $m \circledast A = \{mx, , x \in A\}$, simply we denoted it by mA.

Definition 2.6. (Keisler, 1976; Cutland 1997)

A sequence $\{s_n\}$ convergent to L if and only if $s_N \simeq L$ for all unlimited $N \in \mathbb{N}^*$.

Theorem 2.7. (Pawłowicz, 2013)

A set $A \subseteq \mathbb{R}$ is compact if and only if every sequence in A has a convergent subsequence with limit point in A.

The equivalent nonstandard form: A subset $S \subseteq X$ is said to be compact if for every $x \in S^*$, where S^* is a denotes the nonstandard extension of S, there exists a standard element $y \in S$ such that $x \simeq y$. (Hassan and Hamad, 2021)

Definition 2.8. (Diener and Diener, 1995)

Two Sets A^* and B^* are infinitely close if their **standard parts** coincide:

$$st(A^*) = st(B^*)\,,$$

where $st(A^*) = \{st(x) | x \in A \text{ and } x \text{ limited}\}$. In other words, every point in A^* is infinitely close to some point in B^*, and vice versa.

Definition 2.9. (Bolzano-Weierstrass Theorem)(Robinson, 1996)

Every bounded(limited) sequence has a convergent subsequence.

2.2. Cantor Set in Nonstandard Framework

The classical Cantor set C is constructed iteratively:

i. $C_0 = [0,1]\,,$
ii. $C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right) for\ n \geq 1\,,$
iii. $C = \bigcap_{n=0}^{\infty} C_n\,.$

In nonstandard analysis, we consider the unlimited iteration:

Definition 2.10. (Jumha et al., 2024)

For an unlimited $\omega \in \mathbb{N}^*$, the nonstandard Cantor set is defined as:

$$C^* = \bigcap_{i=1}^{\omega} C_i^*\,,$$

where C_i^* is the nonstandard extension of the i-th iteration of the Cantor set construction.

At the unlimited stage ω, C_ω^* consists of 2^ω closed intervals, each of infinitesimal length $\frac{1}{3^\omega}$.

3. The Nonstandard Cantor Set and Linear Combinations

3.1. Construction of the Nonstandard Cantor Set (Jumha et al., 2024)

The nonstandard Cantor set is constructed analogously to its classical counterpart through iterative removal of central intervals. For each $i \in \mathbb{N}^*$, we define:

$$C_i^* = \bigcup_{n=1}^{2^i} \left[l_i^{(n)}, r_i^{(n)}\right]^*,$$

where $l_i^{(n)} = \frac{a_i^{(n)}}{3^i}$ and $r_i^{(n)} = \frac{a_i^{(n)}+1}{3^i}$, with coefficients $a_i^{(n)} \in \{0,2\}$ determined recursively by:

$$a_0^{(1)} = 0, \quad a_{i+1}^{(n)} = a_i^{(n)}, \quad a_{i+1}^{(2^i+n)} = a_i^{(n)} + 2 \cdot 3^i \text{ for } n = 1,2,\dots,2^i\,.$$

The set C_i^* consists of 2^i pairwise disjoint closed intervals, each of length $\frac{1}{3^i}$. The sequence is decreasing:

$$C_0^* \supset C_1^* \supset C_2^* \supset \cdots$$

By Robinson's lemma, there exists an unlimited $\omega \in N^*$ such that the nonstandard Cantor set is defined as:

$$C^* = \bigcap_{i=1}^{\omega} C_i^* = \left\{\left[0, \frac{1}{3^\omega}\right]^* \cup \cdots \cup \left[1 - \frac{1}{3^\omega}, 1\right]^*\right\}.$$

At the unlimited iteration ω, the set C_ω^* consists of a union of 2^ω closed intervals with infinitesimal lengths $\frac{1}{3^\omega}$. Crucially, we have $C^* \simeq C_\omega^*$ in the sense of Definition 2.8.

3.2. Main Results

Theorem 3.1. A linear Combination of two nonstandard Cantor sets C^* is $[0,2]^*$.

Proof

Let $S^* = C^* \oplus C^* = \{x + y : x, y \in C^*\}$, we demonstrate $S^* = [0,2]^*$ by establishing both inclusions. The inclusion $S^* \subseteq [0,2]^*$ follows immediately from $C^* \subseteq [0,1]^*$.

Since $C^* \subseteq [0,1]^*$, for any $x, y \in C^*$ we have

$$0 \leq x, y \leq 1,$$

and therefore

$$0 \leq x + y \leq 2.$$

Hence,

$$C^* + C^* \subseteq [0,2]^*.$$

For the reverse inclusion $[0,2]^* \subseteq S^*$ requires showing that any $t \in [0,2]^*$ can be expressed as $t = x + y$ for some $x, y \in C^*$. Consider an arbitrary $t \in [0,2]^*$ and examine the following three cases.

Case1: $t \simeq 0^+$ **(positive infinitesimal)**

We must find $x, y \in C^*$ such that $x + y \simeq 0^+$.

At iteration $i = 1$, $C_1^* = \left[0, \frac{1}{3}\right]^* \cup \left[\frac{2}{3}, 1\right]^*$. For $t \simeq 0^+$, observe that the Minkowski sum $C_1^* \oplus C_1^*$ is obtained by summing all combinations of these intervals:

i. $[0, \frac{1}{3}]^* + [0, \frac{1}{3}]^* = [0, \frac{2}{3}]^*$,

ii. $[0, \frac{1}{3}]^* + [\frac{2}{3}, 1]^* = [\frac{2}{3}, \frac{4}{3}]^*$,

iii. $[\frac{2}{3}, 1]^* + [0, \frac{1}{3}]^* = [\frac{2}{3}, \frac{4}{3}]^*$,

iv. $[\frac{2}{3}, 1]^* + [\frac{2}{3}, 1]^* = [\frac{4}{3}, 2]^*$.

Taking the union of these intervals yields:

$$\left[0, \frac{2}{3}\right]^* \cup \left[\frac{2}{3}, \frac{4}{3}\right]^* \cup \left[\frac{4}{3}, 2\right]^*.$$

Since the intervals $\left[\frac{2}{3}, \frac{4}{3}\right]^*$ and $\left[\frac{4}{3}, 2\right]^*$ are adjacent and share the endpoint $\frac{4}{3}$, their union simplifies to $\left[\frac{2}{3}, 2\right]^*$. Therefore,

$$C_1^* \oplus C_1^* = \left[0, \frac{2}{3}\right]^* \cup \left[\frac{2}{3}, 2\right]^*.$$

In particular, the interval $[0, 2/3]^*$ contains t when $t \simeq 0^+$. More precisely, setting $x_1 = y_1 = t/2 \in [0, 1/3]^*$ (since $t/2 \simeq 0^+$ implies $t/2 \in [0, 1/3]^*$ at the first approximation), we obtain $x_1 + y_1 = t$, for $x, y \in C^*$.

For higher iterations $i = 2, 3, \ldots$, we construct sequences $\{x_i\}, \{y_i\}$ with $x_i, y_i \in C_i^*$ and $x_i + y_i = t$ for all i, each sequence composed of infinitesimals close to 0^+.

For each finite $i \in \mathbb{N}$, select $x_i = y_i = t/2$. Since $t \simeq 0^+$, both x_i and y_i are infinitesimal.

By the Bolzano-Weierstrass theorem in nonstandard analysis sense, there exist convergent subsequences $\{x_{i_k}\}$ and $\{y_{i_k}\}$ with limits $c, d \in C^*$ respectively. Then:

$$c \simeq \lim_{k \to \infty} x_{i_k} \simeq 0^+, \quad d \simeq \lim_{k \to \infty} y_{i_k} \simeq 0^+.$$

Thus $t = x_{i_k} + y_{i_k} \simeq c + d \simeq 0^+$, proving $t \in S^*$.

Case2: $t \simeq 2^-$ **(negative infinitesimal from above 2)**

Let $t = 2 - \varepsilon$ with $\varepsilon \simeq 0^+$. By symmetry, we seek $x, y \in C^*$ with $x + y \simeq 2^-$.

For each $i \in \mathbb{N}$, define $x_i = y_i = 1 - \frac{1}{3^i}$. At the unlimited stage ω, we have:

$$x_\omega = y_\omega = 1 - \frac{1}{3^\omega} \simeq 1^-.$$

Thus $x_\omega + y_\omega = 2 - \frac{2}{3^\omega} \simeq 2^- \simeq t$.

By similar arguments to Case 1, using compactness and convergence, there exist $a, b \in C^*$ with $a + b \simeq 2^- \simeq t$, so $t \in S^*$.

Case3: $t = \alpha - \varepsilon$ **for** $\alpha \in (0,2)^*$ **and** $\varepsilon \simeq 0$.

For arbitrary t strictly between 0 and 2, write $t = \alpha - \varepsilon$ where $\alpha \in (0,2)^*$ is the "principal part" and $\varepsilon \simeq 0$.

For each iteration i, we can find $x_i, y_i \in C_i^*$ with $x_i + y_i = t$.

Specifically, setting $x_i = y_i = t/2 \simeq \alpha/2$, the construction ensures $x_i, y_i \in C_i^*$ for all sufficiently large i (this can be verified by examining the structure of C_i^* at each level).

By compactness of C^* and the convergence argument as before, there exist $h, g \in C^*$ such that $h + g \simeq t$. Thus $t \in S^*$.

Conclusion: We have established $[0,2]^* \subseteq S^*$, hence $S^* = [0,2]^*$.

Remark 3.2

Theorem 3.1 extends the classical result of (Pawłowicz, 2013) to the nonstandard setting. The key distinction from the classical proof lies in the use of:

i. unlimited iteration ω,

ii. infinitesimal interval lengths $3^{-\omega}$, and

iii. internal compactness and transfer principles,

which allows us to work directly at the infinitesimal level rather than passing through classical limits.

Theorem 3.3. For $m \in (0,1)^*$, the linear combination $C^* \oplus mC^*$ satisfies:

$$C^* \oplus mC^* = \bigcup_{a\in C^*} [a, a+m]^* \simeq [0,k]^*,$$
$$\text{where } k \simeq a+m.$$

Proof

We present two distinct approaches:

First Approach (Using Transfer Principle):

Applying the Transfer Principle to Pawłowicz's classical result, for each finite $i \in \mathbb{N}$:

$$C_i^* \oplus mC_i^* = \bigcup_{n=1}^{2^i} [l_i^n, r_i^n + m]^* .$$

At the unlimited stage ω, the endpoints l_i^n and r_i^n become infinitesimally close:

$$C_\omega^* \oplus mC_\omega^* = \bigcup_{a\in C_\omega^*} \{[a, a\ +\ m]^* : a \in C_\omega^*\} \simeq [0,k]^* ,$$

where $k \simeq a+m$ for $m \in (0,1)^*$, since the supremum of C_ω^* is infinitely close to C^* and in the same time we have $C_\omega^* \subset C^*$, we conclude that:

$$\begin{aligned} C^* \oplus mC^* &\simeq C_\omega^* \oplus mC_\omega^* \\ &= \bigcup_{a\in C_\omega^*} \{[a, a\ +\ m]^* : a \in C_\omega^*\} \\ &\simeq [0,k]^* . \end{aligned}$$

Thus

$$C^* \oplus mC^* = \bigcup_{a\in C^*} [a, a+m]^* \simeq [0,k]^* ,$$

for $k \simeq m+a$ and $m \in (0,1)^*$.

Second Approach (Via Interval Decomposition):

Recall $C_\omega^* = \left[0, \frac{1}{3^\omega}\right]^* \cup \dots \cup \left[1-\frac{1}{3^\omega}, 1\right]^*$ with $C^* \simeq C_\omega^*$. Using Definition 2.5 and basic interval arithmetic:

$$\left[0, \frac{1}{3^\omega}\right]^* + m\left[0, \frac{1}{3^\omega}\right]^* = \left[0, \frac{1+m}{3^\omega}\right]^* ,$$

$$\left[0, \frac{1}{3^\omega}\right]^* + m\left[1-\frac{1}{3^\omega}, 1\right]^* = \left[m - \frac{m}{3^\omega}, \frac{1}{3^\omega} + m\right]^* ,$$

$$\vdots$$

$$\begin{aligned} &\left[1-\frac{1}{3^\omega}, 1\right]^* + m\left[1-\frac{1}{3^\omega}, 1\right]^* \\ &\qquad = \left[1+m-\frac{1+m}{3^\omega}, 1+m\right]^* . \end{aligned}$$

Taking the union over all such intervals and noting that infinitesimal gaps become negligible:

$$\begin{aligned} C_\omega^* \oplus mC_\omega^* = \left[0, \frac{1+m}{3^\omega}\right]^* &\cup \cdots \\ &\cup \left[1+m-\frac{1+m}{3^\omega}, 1+m\right]^* . \end{aligned}$$

Since the endpoints of these intervals are infinitely close to those of $[0,1+m]^*$, and by Definition 2.8:

$$C_\omega^* \oplus mC_\omega^* \simeq [0,1+m]^* .$$

With the same last argument of the first method about the relation between C_ω^* and C^*, we conclude:

$$C^* \oplus mC^* = \bigcup_{a\in C^*} [a, a+m]^* \simeq [0,1+m]^* = [0,k]^* ,$$

for $k = m+1$ and $m \in (0,1)^*$, where $m \simeq 1^-$. Thus, $m \simeq 1 - \frac{1}{3^\omega}$. Hence $k \simeq 2^-$. More precisely, we establish that

$$C^* + mC^* \simeq [0,1+m]^*,$$

meaning that both sets have the same standard part:

$$\text{st}\,(C^* + mC^*) = [0,1+m].$$

At the unlimited iteration stage, the set C^* consists of 2^ω disjoint intervals of infinitesimal length $3^{-\omega}$. Consequently, the sum $C^* + mC^*$ is an internal union of finitely (in the nonstandard sense) many intervals whose endpoints differ from those of $[0,1+m]^*$ by at most infinitesimals. The “gaps” that persist in this union are themselves of infinitesimal magnitude.

Corollary 3.4. For any $a, b \in (\mathbb{R}^*)^+$, the general linear combination satisfies:

$$aC^* \oplus bC^* = [0, a+b]^* .$$

Proof

This follows by scaling arguments from Theorem 3.1. Since $C^* \subseteq [0,1]^*$, we have:

$aC^* \subseteq [0, a]^*$,

$bC^* \subseteq [0, b]^*$,

Therefore $aC * \bigoplus bC^* \subseteq [0, a + b] *$.

For the reverse inclusion, given any $t \in [0, a + b]^*$, we can write:

$$t = a \cdot \left(\frac{t}{a+b}\right) + b \cdot \left(\left(\frac{t}{a+b}\right)\right).$$

Since $\left(\frac{t}{a+b}\right) \in [0,1]^*$ and by the density of $C^* \in [0,1]^*$ (in the nonstandard sense), there exist $c_1, c_2 \in C *$ such that $c_1 \simeq c_2 \simeq \frac{t}{a+b}$. Therefore: $t \simeq ac_1 + bc_2 \in aC^* \bigoplus bC^*$, which completes the proof.

Theorem 3.5. The Lebesgue measure μ of the linear combination $C^* \bigoplus mC^*$ satisfies:

$$\mu(C^* \bigoplus mC^*) \simeq (1 + m) - \sum_{n=0}^{\omega} \frac{2^n m}{3^{n+1}} ,$$

for $m \in (0,1)^*$.

Proof

The measure evolves recursively through the construction stages. At each finite iteration n, the removed intervals have total length $\frac{2^n m}{3^{n+1}}$. Transferring this to the nonstandard framework and summing over all n from 0 to ω yields the result.

Corollary 3.6. The nonstandard measure of linear combinations satisfies:

i. $\mu * (C^* \bigoplus C^*) = 2$,

ii. $\mu * (C^* \bigoplus mC^*) \simeq 1 + m$ for $m \in (0,1)^*$,

iii. $\mu * (aC^* \bigoplus bC^*) \simeq a + b$ for $a, b \in (0, \infty)^*$.

Proof

This follows directly from Theorems 3.3 and 3.5.

Theorem 3.7. For distinct $m_1, m_2 \in (0,1)^*$ with $m1 \neq m2$, the symmetric difference:

$$(C^* \bigoplus m_1 C^*) \triangle (C^* \bigoplus m_2 C^*) ,$$

contains nonempty internal open sets in $\mathbb{R}^*$.

Proof

Recall that in nonstandard analysis, an internal set is a set that can be described by a formula that does not use the standard predicate (or is transferred from a standard set). An open set in $\mathbb{R}^*$ is a set that contains a neighborhood around each of its points.

We want to find an interval (which is internal and open in $\mathbb{R}^*$) that is contained in one of the sets $C^* \bigoplus m_1 C^*$ but not in the other, or vice versa. Consider the structure of $C^* \bigoplus mC^*$.

From Theorem 3.3, we have: $C^* \bigoplus mC^* \simeq [0, k]^*$, where $k = 1 + m$. However, note that $C^* \bigoplus mC^*$ is not exactly an interval but a union of many small intervals (at the infinitesimal scale). The symmetric difference between two such sets for different m should contain gaps that are internal open sets. The following steps outline how to identify these gaps:

Step 1: Without loss of generality, assume $m_1 < m_2$. Then by Theorem 3.3

$k_1 = 1 + m_1$ and $k_2 = 1 + m_2$ with $k_1 < k_2$.

Step 2: Consider the set $(C^* \bigoplus m_2 C^*) \setminus (C^* \bigoplus m_1 C^*)$.

We aim to show this set contains an internal open interval $(a, b)^*$ with $a < b$ in $\mathbb{R}^*$.

Step 3: Recall that C^* is the nonstandard Cantor set, which is closed and nowhere dense in the standard sense, but in the nonstandard sense, it is a union 2^ω intervals of infinitesimal length. The linear combinations $C^* \bigoplus mC^*$ are also unions of intervals, preserves this fractal structure but with modified gap distributions depending on m. The gaps between these intervals are also infinitesimal.

Step 4: Using the transfer principle, we can consider the standard Cantor set C and its linear combinations. It is known that for standard $m_1 \neq m_2$, the symmetric difference

$$(C \bigoplus m_1 C) \triangle (C \bigoplus m_2 C) ,$$

contains nonempty open intervals. This occurs because the Cantor set's self-similar structure produces different gap patterns for different linear combination parameters.

Step 5: By the Transfer Principle of nonstandard analysis, this result extends to the nonstandard framework. Therefore, for $m_1, m_2 \in (0,1)^*$ with $m_1 \neq m_2$, there exists an internal open in $\mathbb{R}^*$.

Step 6: More concretely, we can look at the endpoints of the intervals in $C^* \oplus mC^*$. For example, at some level of the construction, the set $C^* \oplus mC^*$ will have a gap that is not present in the other set. This gap, being an interval, is an internal open set.

Step 7: To be more precise, we can use the fact that the Cantor set $C^*$is the intersection of the sets C_n^*. Then $C^* \oplus mC^*$ is the intersection of $C_n^* \oplus mC_n^*$. Each $C_n^* \oplus mC_n^*$ is a union of closed intervals. For large n (even unlimited ω), the gaps between these intervals become very small, but for different m, the gaps are in different positions.

Step 8: We can find a particular gap $C_n^* \oplus m_1 C_n^*$ that is not a gap in $C_n^* \oplus m_2 C_n^*$ for some n. Then, by the nature of the construction, this gap will persist in the limit (for the nonstandard set) and will be an internal open set.

We can construct such an internal open set explicitly by examining finite construction levels as follows:

At level $n = 1$, we have:

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],$$

$$C_1 \oplus mC_1 = \left[0, \frac{1+m}{3}\right] \cup \left[\frac{2m}{3}, \frac{1}{3} + m\right] \cup \left[\frac{2}{3}, 1 + \frac{m}{3}\right] \cup \left[\frac{2}{3}(1+m), 1+m\right].$$

Consider the gap between the first two intervals:

$$J(m) = \left(\frac{1+m}{3}, \frac{2m}{3}\right).$$

This gap is nonempty when $\frac{1+m}{3} < \frac{2m}{3}$, when $m > 1$, which never occurs for $m \in (0,1)$.

On the other hand, $\left(\frac{1+3m}{3}, \frac{2}{3}\right)$ is define a gap between the first two intervals for $0 < m < \frac{1}{3}$. So, for $m < \frac{1}{3}$, there is a gap $J(m)$ in $C_1 + mC_1$. Now, take $m_1 < \frac{1}{3}$ and $m_2 \geq \frac{1}{3}$. Then:

i. $J(m_1)$ is a gap in $C_1 + m_1 C_1$, so $J(m_1) \cap (C_1 + m_1 C_1) = \emptyset$.

ii. For $m_2 \geq \frac{1}{3}$, the gap $J(m_1)$ might not be a gap in $(C_1 + m_2 C_1)$.

Thus, at each level n, the set $C_n + mC_n$ contains gaps between these intervals. The positions and sizes of these gaps depend on the parameter m. Hence, for distinct $m_1, m_2 \in (0,1)$, there exists some finite level N and an interval $J \subset [0, 1+m]$ such that:

i. J is a gap in $C_N \oplus m_1 C_N (i.e., J \cap (C_N \oplus m_1 C_N) = \emptyset)$.

ii. J is not a gap in $C_N \oplus m_2 C_N (i.e., J \cap (C_N \oplus m_2 C_N) \neq \emptyset)$.

Let $J(m_1)$ denotes to the gap with first property. Then $J(m_1)$ persists in all subsequent construction levels. That is, for all $n \geq 1$:

i. $J(m_1) \cap (C_n \oplus m_1 C_n) = \emptyset$.

ii. $J(m_1) \cap (C_n \oplus m_2 C_n) \neq \emptyset$.

This is because the Cantor set construction is monotonic: $C_{n+1} \subset C_n$, so $C_{n+1} \oplus mC_{n+1} \subset C_n \oplus mC_n$. Once a gap appears at some level, it remains in all higher levels. Now, By the transfer principle, for nonstandard $m_1, m_2 \in (0,1)^*$ with $m_1 \neq m_2$, there exists an unlimited $\omega \in \mathbb{N}^*$ and an internal open interval $J^* \subset \mathbb{R}^*$ such that:

i. $J^* \cap (C_\omega^* \oplus m_1 C_\omega^*) = \emptyset$.

ii. $J^* \cap (C_\omega^* \oplus m_2 C_\omega^*) \neq \emptyset$.

Since

$$C^* \oplus mC^* = \bigcap_{n=1}^{\omega} (C_n^* \oplus mC_n^*),$$

we have:

$J^* \cap (C^* \oplus m_1 C^*) = \emptyset$.

$J^* \cap (C^* \oplus m_2 C^*) \neq \emptyset$.

Therefore, $J^* \subset (C^* \oplus m_1 C^*) \triangle (C^* \oplus m_2 C^*)$.

The set J^* is internal because it is the transfer of a standard open interval. It is open in the nonstandard topology because it satisfies:

$$\forall x \in J^*, \exists \epsilon > 0, (x-\epsilon, x+\epsilon)^* \subset J^*,$$

where $(x-\epsilon, x+\epsilon)^*$ is the nonstandard extension of the standard open interval.

Let $J = (a,b)$ be a standard open interval. Then $J^* = \{x \in \mathbb{R}^* \mid a < x < b\}$.

Take $x \in J^*$. This means that $a < x < b$, where these inequalities are in $\mathbb{R}^*$. We need to find a standard $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon)^* \subset J$.

i. Since $x > a$, the distance $x - a$ is a positive hyperreal number.

ii. Since $x < b$, the distance $b - x$ is also a positive hyperreal number.

Now, choose a standard ϵ to be any positive real number that is smaller than both of these hyperreal distances. For example, define

$$d_1 = x - a (a \text{ positive hyperreal}),$$

$$d_2 = b - x (a \text{ positive hyperreal}).$$

We can simply choose $\epsilon = \frac{1}{2} min(st(d_1), st(d_2))$, where "$st$" denotes the standard part function. If d_1 or d_2 is unlimited or appreciable in $\mathbb{R}^*$, then any standard ϵ will work. The key point is that a standard, positive ϵ exists. Why does this ϵ work?

By construction, $y \in (x-\epsilon, x+\epsilon)^*$ then $x - \epsilon < y < x + \epsilon$.

i. From $y > x - \epsilon$ and $x > a + \epsilon$ (since $\epsilon < x - a$), it follows that $y > (a+\epsilon) - \epsilon = a$.

ii. From $y < x + \epsilon$ and $x < b - \epsilon$ (since $\epsilon < b - x$), it follows that $y < (b - \epsilon) + \epsilon = b$.

Therefore, $a < y < b$, which means $y \in J^*$. We have shown that any y in the small nonstandard interval also belongs to J^*, so $(x-\epsilon, x+\epsilon)^* \subset J^*$.

i. J^* is internal because it is the direct transfer of the standard set (a,b).

ii. It is open in the nonstandard topology because for any point $x \in J^*$, one can use the distance from x to the endpoints a and b (via $st(x-a)$ and $st(b-x)$) to define a standard-radius nonstandard interval around x that remains entirely within J^*. This is the nonstandard analogue of the classical proof that an open interval is open.

This follows from the fact that the Cantor set construction removes different middle portions depending on m, and for distinct value of m, the resulting gap structures different.

Corollary 3.8. Let C^* denote the nonstandard Cantor set in [0,1]. Then the Minkowski sum $C^* \oplus C^*$ contains an infinitesimal neighbourhood of [0,2]. In particular, for every standard $x \in [0,2]$, there exist $y, z \in C^*$ such that $x \simeq y + z$.

Proof

In the classical setting, it is known that the Minkowski sum $C \oplus C = [0,2]$, where C is the standard Cantor set. In the nonstandard setting, the construction of C^* involves intervals of infinitesimal length at unlimited iteration stages. Since every element of C is infinitely close to some element of C^*, the Minkowski sum $C^* \oplus C^*$ produces intervals whose endpoints approximate those of [0,2] up to infinitesimals. Hence, for every standard $x \in [0,2]$, there exist $y, z \in C^*$ such that $x \simeq y + z$. Therefore, $C^* \oplus C^*$ fills an infinitesimal neighbourhood of [0,2].

4. Applications of Nonstandard Cantor Set Theory in Nanoscience

The nonstandard analysis framework developed in this paper, particularly our results on linear combinations of Cantor sets, provide powerful mathematical tools for addressing fundamental challenges in nanoscience and nanotechnology. The ability to rigorously handle *infinitesimal scales* and *multi-scale phenomena* through nonstandard analysis offers unique advantages for modelling complex nanoscale systems where discrete atomic behavior transitions to continuum material properties (Chen and Lv, 2022; Bordonhos et al., 2023; Capone et al., 2024). The authors have shown that for $m \in (0,1)^*$, $C^* + mC^* \simeq [0,1+m]^*$, in the sense of infante closeness, while internally the set remains a union of 2^ω intervals of infinitesimal length. Our main theoretical results establishing that $C^* \oplus C^* = [0,2]^*$ and characterizing $C^* \oplus mC^*$ for $m \in (0,1)^*$ have direct analogues in nanoscale systems where linear combinations of atomic orbitals, molecular structures, or material phases determine emergent properties. The nonstandard framework allows us to mathematically capture the transitional behaviors that occur at the nanoscale, where quantum effects dominate but classical approximation begin to emerge.

4.1. Infinitesimal Structure and Multi-Scale Decomposition

At an unlimited iteration stage, the nonstandard Cantor set C^* consists of an internal family of intervals of length $3^{-\omega}$, which are infinitesimal. The linear combination $C^* + mC^*$ therefore produces a new internal set composed of sums of such infinitesimal intervals.

From a structural perspective, this can be interpreted as a multi-scale decomposition:

i. At the *microscopic level*, the set is highly fragmented, consisting of disjoint infinitesimal components.

ii. At the *macroscopic level*, the same set is infinitely close to a connected interval.

Thus, the results demonstrate that a set may simultaneously exhibit:

i. local discreteness (*fragmentation*), and

ii. global continuity (*interval behavior*).

This coexistence provides a rigorous mathematical description of multi-scale phenomena within a single framework

The statement "a molecular orbital can be expressed as a linear combination of atomic orbitals" is a well-defined first-order statement in standard mathematics and quantum physics (Parrish et al., 2021). By the Transfer Principle, this statement must also hold true in the nonstandard framework. Therefore, there exists a nonstandard molecular orbital ψ^* that is a linear combination of the nonstandard atomic orbitals ϕ_i^*:

$$\psi^* = \sum_{i,j} m_{ij}^* \phi_i^* \oplus \phi_j^*.$$

Each electron orbitals assumed to be considered as a circular Cantor like sets in nonstandard sense C_ω^*, each electron position assumed to be an infinitesimal interval part of C_ω^*. A collection of electron orbital levels(atom) assumed to be a family of circular Cantor like sets, rather than nuclear, see Figure 1. A linear combination of orbitally arranged members of this family is considered a molecular orbital. An orbital represents an energy level where electrons reside. These energy levels are located at specific distances from the nucleus, and each level can accommodate a fixed number of electrons (Santos et al., 2025).

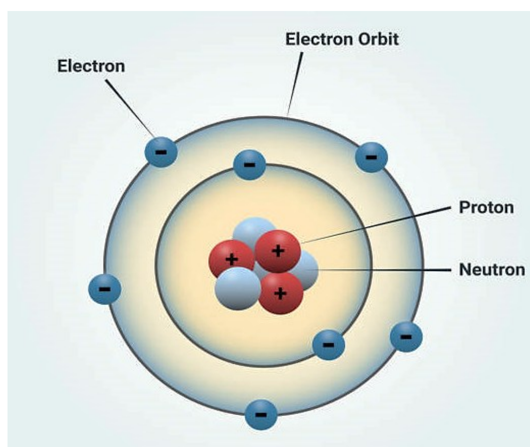

Figure 1. The electron orbitals assumed to be considered as a circular Cantor like sets.

4.2. Direct Applications to Linear Combination of Atomic Orbitals (LCAO)

The LCAO theory, fundamental to computational chemistry and materials science, describes molecular orbitals as linear combinations of atomic orbitals. Our work on nonstandard linear combinations provides a rigorous mathematical framework for extending LCAO theory to nanoscale systems where traditional approximations break down (Santos et al., 2025).

In a crystal, the potential energy resulting from all the atoms within the lattice is depicted in Figure 3 (Wei et al., 2022).

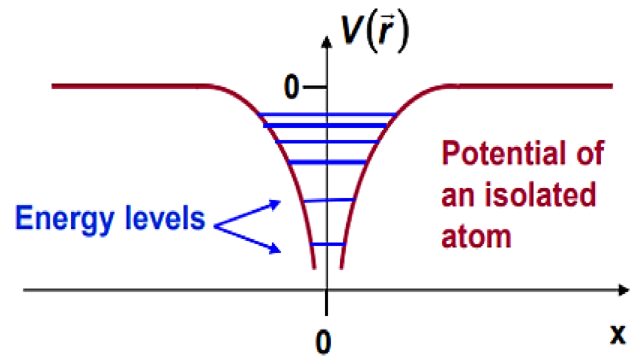

Figure 2. The potential energy of an electron.

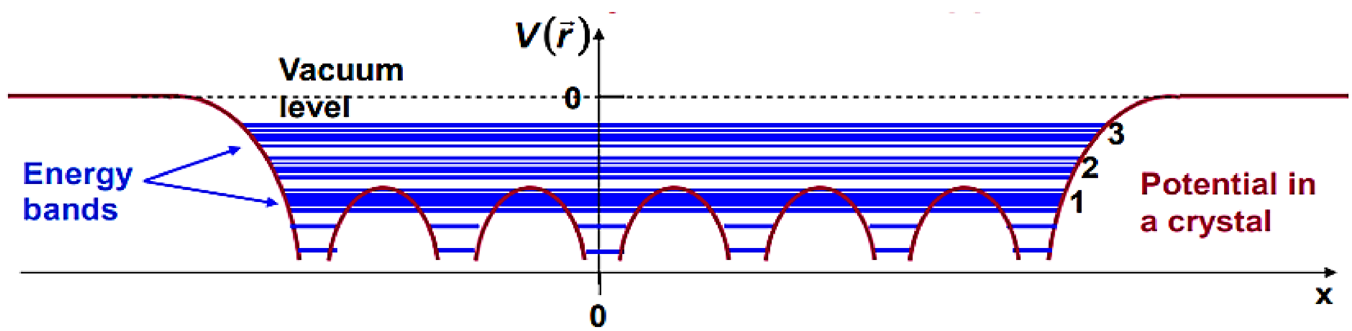

Figure 3. The potential energy of a crystal.

When an atom becomes part of a crystalline lattice, its lowest electron energy levels and associated wavefunctions remain largely unchanged. By contrast, the higher-energy levels, which are typically linked to the outer atomic shells, undergo significant modification: their wavefunctions lose localization and spread across the entire crystal (Parrish et al., 2021; Wei et al., 2022).

For bands at higher energies (such as Bands 2 and 3 in the figure), the periodic atomic potential may be regarded as a weak perturbation, and the nearly free-electron model yields a reliable description. However, for lower-energy bands (such as Band 1), this periodic potential constitutes a strong perturbation, limiting the effectiveness of the nearly free-electron approximation. Motivated by the observation that the low-energy band displays feature reminiscent of Cantor-type structures, we suggest that tools from nonstandard analysis may allow these strong perturbative effects to be treated as infinitesimal, or even micro-infinitesimal, contributions.

5. Discussions

This work shows that the nonstandard Minkowski sum $C^* + mC^*$ exhibits a dual nature: it is globally infinitely close to the interval $[0,1+m]$, while internally remaining a fragmented union of infinitesimal intervals. This reveals a form of hidden continuity, where fractal gaps persist at the micro-level but disappear at the macroscopic scale. The nonstandard framework allows direct analysis at infinitesimal resolution, clarifying how gap structures depend on m and avoiding classical limiting arguments.

6. Conclusions

We extended classical results on Cantor set sums to the nonstandard setting, proving that $C^* + mC^* \simeq$

$[0,1+m]$ while preserving infinitesimal structure. This approach provides a unified view of local fractal behavior and global interval formation. The results confirm the strength of nonstandard analysis in studying fractals and open directions for further research in multiscale and applied contexts.

Conflict of interest

The authors declare that there are no conflicts of interest pertaining to this manuscript.

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