Sulaimani Journal for Pure and Applied Sciences Vol. 27 Issue 2 (2025) 12-22.

Journal homepage https://sjpas.univsul.edu.iq

Sulaimani Journal for Pure and Applied Sciences

ISSN:1812-4100

Original Article

New Global Interchange Formula of Parameter Conjugates Gradient Method for Solving Optimization Problems

Ivan Subhi Latif [a], Lajan Jalil Mohammed [b,*] and Bahar Obed Ali [c]

[a] Department of Mathematics, College of Education, Salahaddin University-Erbil, Erbil, Kurdistan Reign, Iraq. University of Salahaddin, Erbil, Kurdistan Region, Iraq.

[b] Department of Information Technology and Computer Science, College of Science, Salahaddin University-Erbil, Erbil, Kurdistan Region, Iraq.

[c] Department of Mathematics, College of Science, Salahaddin University-Erbil, Erbil, Kurdistan Region, Iraq.

ARTICLE INFO

Article History:
Received: 12/06/2025
Revised: 9/08/2025
Accepted: 20/09/2025
Published online: 25/12/2025

Key Words:
Unconstrained optimization
Hybrid conjugate gradient method
Inexact line search
Global convergence

OPEN ACCESS

ABSTRACT

In this paper, we propose a new global interchange formula of a parameter in the conjugate gradient (CG) method designed specifically for unconstrained nonlinear optimization problems. The main contribution of this study lies in the introduction of a novel scalar interchange parameter that guarantees the generation of a descent search direction at each iteration without relying heavily on the line-search conditions. This improvement not only strengthens the global convergence properties of the algorithm but also enhances its numerical stability. To validate the effectiveness of the proposed method, we provide a comprehensive comparison with several classical CG algorithms, including Hestenes–Stiefel (HS), Liu–Storey (LS), Fletcher–Reeves (FR), and Polak–Ribiere–Polyak (PRP). Extensive numerical experiments on a wide range of benchmark test functions demonstrate that the proposed algorithm consistently outperforms existing methods in terms of the number of iterations (NOI), the number of function evaluations (NOF), and robustness across different problem dimensions. The results confirm that the proposed method is a reliable and efficient tool for solving large-scale unconstrained optimization problems, offering both theoretical novelty and practical significance to the field optimization literature.

1. Introduction

In the the focus of this paper is the study of unconstrained nonlinear optimization problem which plays a central role in applied mathematics, machine learning, engineering design, and scientific computing.

Where the problem $F: R^n \to R$ is continuously differentiable function in R^n and R^n is an n-dimensional Euclidean space and is may be very large.

$$\min f(x), x \in R^n. \quad (1.1)$$

The gradient of $f(x)$ is denoted by g. Conjugate gradient methods are highly efficient for solving this type of problem, because they do not require explicit matrix storage and are highly suitable for large-scale problems. Starting from an initial point x_0, CG methods construct a sequence of iterates x_k using the update rule:

$$x_{k+1} = x_k + \alpha_k d_k \ , \quad k = 0,1,2, ... \quad (1.2)$$

* Corresponding author.
E-mail address: lajan.mohammed@su.edu.krd (L. Mohammed)

Where the step length is determined by a line search direction (Nocedal & Wright, 2006; Sulaiman & Hassan, 2023) and d_k is the search direction which is determined by:

$$d_k = \begin{cases} -g_k & , \text{if } k = 0 \\ -g_k + \beta_k d_{k-1} & , \text{if } k > 0 \end{cases}. \quad (1.3)$$

Where g_k is gradient for objective function at iteration k at x_k and β_k is a scalar parameter.

Several well-known formulas for β_k include:

$$\beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \quad (1.4) \text{ Fletcher Reeves (FR).}$$

$$\beta_k^{CD} = \frac{-\|g_k\|^2}{d_{k-1}^T g_{k-1}}, \quad (1.5) \text{ The conjugate descent (CD).}$$

$$\beta_k^{DY} = \frac{\|g_k\|^2}{d_k^T y_{k-1}}, \quad (1.6) \text{ Dai-Yuan (DY).}$$

$$\beta_k^{HS} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \quad (1.7) \text{ Hestenes-Stiefel (HS).}$$

$$\beta_k^{PRP} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2}, \quad (1.8) \text{ Polak-Ribiere-Polak PRP.}$$

$$\beta_k^{LS} = \frac{-g_k^T y_{k-1}}{d_{k-1}^T g_{k-1}}, \quad (1.9) \text{ Liu-Storey (LS).}$$

Here, the Euclidean norm is denoted by $\|\cdot\|$.

Where

$$y_{k-1} = g_k - g_{k-1}. \quad (1.10)$$

We see from formula (1.2) that for each $k \geq 1$, the directional derivative of f at x_k along direction d_k is given by:

$$g_{k-1}^T d_k = -\|g_k\|^2 + \beta_k g_k^T d_{k-1}. \quad (1.11)$$

It is clear that if exact line search is used, then we have for any $k \geq 0$

$$g_k^T d_k = -\|g_k\|^2 < 0. \quad (1.12)$$

Consequently, vector d_k is a descent direction of f at x_k, (Zoutendijk, 1970) and proved the global convergence of FR method has been further analysed in terms of convergence behaviour and descent properties under line search conditions (Latif & Al-baali, 2022; Latif, 2014; Nocedal & Wright, 2006; Hassan & Alashoor, 2017; Abbo & Hassan, 2023; Sulaiman et al., 2020; Mohammed, & Latif, 2022; Labed & Aounallah, 2023). The global convergence of algorithms related to the FR method with strong Wolfe condition (Sadraddin & Latif, 2024) also proved that FR is a superior method compared to others. The global convergence of PR, LS, and HS has not been established yet, the main reason is that it cannot guarantee the descent in objective function values at each iteration (Ibrahim & Mohammed, 2022; Liu et al., 2020), for further reading and recent finding of cg methods refer to (Birgin & Martinez, 2001).

A key factor of global convergence is selecting the step size α_k. The most common search is exact line search: (Yuan et al., 2009).

$$f(x_k + \alpha_k d_k) = \min_{\alpha \geq 0} f(x_k + \alpha d_k). \quad (1.13)$$

We propose new hybrid cg method with a simple formula for interchange scalar parameter β_k which generates a descent direction and satisfies the objective function at each iteration (Fletcher, 2000). Our new algorithm is compared with Hestenes-Stiefel and Liu-Storey methods (Hestenes & Stiefel, 1952; Nakamura et al., 2013; Liu & Storey, 1991). Section 2 presents the algorithms.

Section 3 establishes the global convergence of the new hybrid method; Impractical exact line search is often computationally expensive therefore, and in exact line search strategies such as the Armijo rule are widely adopted.

Finally, we report numerical results to test the proposed methods using test problems from the (Nocedal & Wright, 2006).

2. Conjugate gradient algorithm (CG)

Numerical methods for solving optimization problems such as formula (1.1) are numerous, and the conjugate gradient method stands out for nonlinear unconstrained optimization, because it does not require any matrix computations. This study focuses on the global convergence of the conjugate gradient method for nonlinear optimization with inexact Armijo line search, conducting an exact line search is often impractical due to the high cost of excessive Function evaluations, even when terminated with a small accuracy tolerance $\varepsilon > 0$.

Compromising on accuracy can hinder the algorithm's overall convergence. However, using a line search that ensures a sufficient degree of accuracy or descent in the function value can promote the overall convergence of the algorithm, the definition of an acceptable step length known as Armijo's rule:

Set scalars S_k , β, μ_1 , $L > 0$, with $S_k = -g_k^T d_k / (L \|d_k\|^2)$, $\mu_1 \in \left(0, \frac{1}{2}\right)$, $\beta \in (0,1)$. Let α_k be the largest α_k in $\{S, S\beta, S\beta^2\}$such that:

$$f(x_k + \alpha_k d_k) \le f(x_k) + \mu_1 \alpha_k g_k^t d_k. \tag{2.1}$$

Choosing the parameter β is crucial for the implementation of the line search. If β is too large, the process may be too slow, else if β is too small, the process may be too fast and lose the optimal step size. Hence, a suitable step size must be chosen at each iteration.

3. Outline of the Hastens - Stiefel conjugate gradient (HSCG) algorithm

Step 1: Given a starting point and a tolerance $\varepsilon = 1 \times 10^{-6}$,
Step 2: Set $k = 1$, $d_0 = -g_0$,
Step 3: Terminate if $\|g_k\| \le \varepsilon$ and stop, otherwise $k = 1$ go to step (4),
Step 4: Compute x_k used formula (1.2) and obtain α_k from line search procedure satisfy formula (2.1).
Step 5: Find the new direction using formulas (1.3) and (1.7).
Step 6: Go to step (3).

4. Outline of the Liu - Storey conjugate gradient (LSCG) algorithm

Step 1: Given a starting point and tolerance $\varepsilon = 1 \times 10^{-6}$,
Step 2: Set $k = 1, d_0 = -g_0$.
Step 3: Terminate if $\|g_k\| \le \varepsilon$, then stop, otherwise, go to step 4.
Step 4: Compute x_k using formula (1.2) and obtain α_k from line search procedure satisfying formula (2.1).
Step 5: Find the new direction using formulas (1.3) and (1.9).

Step 6: Go to step 3.

5. New hybrid conjugate gradient method $\left(\beta_k^{ILB}\right)$

In Hastens – Stiefel and Liu-Storey conjugate gradient methods (Hastens & Stiefel, 1952; Abubakar et al., 2022) generating a suboptimal direction and a small step from x_{k-1} to x_k can result in subsequent directions and steps that are also suboptimal. To address this, a restart along the gradient method may be necessary to correct the optimization path.The next direction d_k and the next steps $\alpha_k d_k$ are also likely to be poor. Unless a restart along the gradient methods is performed.

To establishing the convergence results of nonlinear conjugate gradient methods, it is usually required that the step size α_k be obtained using Armijo rule (Shetty, 1993). The direction search satisfies a simple formula for interchange scalar β_k, which defined as

$$\beta_k^{ILB} = \min \left\{ \begin{array}{lll} Q_k \dfrac{g_k^T\, y_{k-1}}{d_{k-1}^T y_{k-1}} & \text{if} & \dfrac{g_k^T\, g_{k-1}}{d_{k-1}^T\, y_{k-1}} \le \dfrac{\|g_k\|}{d_{k-1}^T y_{k-1}} \\ (1 - Q_k) \dfrac{-g_k^T\, y_{k-1}}{d_{k-1}^T y_{k-1}} & \text{if} & \dfrac{g_k^T g_{k-1}}{d_{k-1}^T\, y_{k-1}} > \dfrac{\|g_k\|}{d_{k-1}^T y_{k-1}} \end{array} \right\} . \tag{3.1}$$

Where $0 \le Q_k \le 1$.

If $Q_k = 0$ then we have a special case $\beta_k^{ILB} = \beta_k^{LS}$ and if $Q_k = 1$ then we have a special case $\beta_k^{ILB} = \beta_k^{HS}$. In our implementations, we used the following parameter settings $\varepsilon = 10^{-4}$ which sufficient decrease while avoiding overly conservative steps. A back tracking strategy with reduction factor $\beta \in (0,1)$ was applied, we selected $\beta = 0.5$ to balance efficiency and convergence. The initial trial step size was set to $\propto_0 = 1$. These choices balance computational cost and stability, allowing the proposed method to maintain convergence while reducing the number of unction evaluations.

6. Outline of the new hybrid conjugate gradient method (β_k^{ILB})

Step 1: Let x_0 be an initial point to minimize of f ,
Step 2: Set $k = 1$ and $d_k = -g_k$,
Step 3: Do a line search using formula (1.2), where $\alpha_k$is computed by 2.1 and $d_k$is computed used (1.3), (3.1),
Step 4: If $\|g_{k+1}\| \leq \varepsilon$, set $x^* = x_k$ and stop.
Step 5: If $k + 1 > n$ or when ever $d_{k+1}^T\ g_{k+1} > -0.8\ \|g_{k+1}\|^2$ is satisfied then set $k = k + 1$and go to step (3),
Step 6: Set $x_{k+1} = x_1$ and go to step 2.

7. The convergence analysis of the new hybrid conjugate gradient algorithm (β_k^{ILB})

In the following section we consider the convergence of the proposed algorithm. Assume that the Assumption Holds (Beale, 1988; Nocedal & Wright, 2006):

(H1) The level set

$$L = \{x \in R^n : f(x) \leq f(x_0)\}. \quad (3.2)$$

Is bounded, and x_0 is an initial point.

(H2) In some neighbourhood U of L, if f is continuously differentiable and its gradient is Lipschitz continuous, namely there exist a constant $L_1 > 0$ such that:

$$\|g(x) - g(y)\| \leq L_1 \|x - y\|;\ \forall\, x, y \in U. \quad (3.3)$$

It follows from the inequality 3.3 that $\{f(x_k)\}$ is decreasing, so that the sequence $\{x_k\}$ is contained in U . We can get from assumption (H1), (H2) that there exists a positive constant $\beta > 0$ and $\gamma_1 > 0$ such that

$$\|x\| \leq \beta \quad > \quad \|g_k\| \leq \gamma_1 \quad \forall x \in U. \quad (3.4)$$

Lemma: $\beta_k > 0$ in algorithm 3.1 and if $0 < g_k^T g_{k-1} < \|g_k\|^2$, then $\beta_k^{LS} < \beta_K^{CD}$.

Proof: From 1.7 we have

$$g_k^T d_{k-1} = g_{k-1}^T d_{k-1} \geq (\delta_1 - 1)^2 g_{k-1}^T d_{k-1}.$$

Since $\delta_1 < 1$, it clearly implies that $\delta_1 - 1 < 0$.

Theorem: In the algorithm 3.1 assume that $\boldsymbol{\alpha_k}$is determined by the Armijo line search from Equation 2.1. If the direction, then the step size satisfies the necessary condition for ensuring convergence.

$$g_k^T\, d_k < 0. \quad (3.5)$$

Proof: We establish the validity of formula (3.5) by examining the HS and LS hybrid conjugate gradient algorithm $d_{k+1}$is defined by Equation 3.1. We consider two distinct cases:

First case: If $k = 1$ and $d_k = g_k$then

$$g_k^T\, d_k = g_k^T(-g_k) = -\|g_k\|^2 < 0. \quad (3.6)$$

Second case: If $k > 1$

$$d_k = -\left(1 + \beta_{k-1}^{ILB} \frac{g_k^T\, d_{k-1}}{\|g_k\|^2}\right) g_k + \beta_{k-1}^{ILB} d_{k-1}. \quad (3.7)$$

Then we have

$$g_k^T d_k = -\|g_k\|^2 - \left(\beta_{k-1}^{ILB} \frac{g_k^T\, d_{k-1}}{\|g_k\|^2}\right) g_k^T g_k + \beta_{k-1}^{ILB} g_k^T d_{k-1} = -\|g_k\|^2 < 0. \quad (3.8)$$

Theorem: Let assumptions H1 and H2 hold, and consider the new hybrid conjugate gradient method defined in Algorithm 3.1, where the search direction is a descent direction and the step size is determined by the Armijo line search as outlined in Equation 2.1. Under these conditions, the following result holds:

$$\lim_{k \to \infty} \inf \|g_k\| = 0.$$

Proof: To prove this by contradiction, assume that the conclusion is false. Then, there exists a constant $\varepsilon > 0$ such that $\|g_k\| \geq \varepsilon \quad \forall k$, according to Equation 3.8, we obtain...

$$\|d_k\|^2 = \beta_{k-1}^{ILB} \ \|d_{k-1}\|^2 - 2h_{k-1}\ d_{k-1}^T g_{k-1} - h_{k-1}^2 \|g_{k-1}\|^2. \quad (3.9)$$

$$h_{k-1} = 1 + \beta_{k-1}^{ILB} \frac{g_{k-1}^T d_k-1}{\|g_{k-1}\|^2}. \quad (3.10)$$

Divided in to two cases as follows

Case 1: According to Algorithm 3.2, If $g_k^T \ g_{k-1} > 0$. We have $\beta_k^{ILB} < \beta_k^{FR}$ based on the LS and HS formulas by the use of theorem3.2.1 and the same argument of theorem 2.2 in (Zhang & Zhou, 2008).

Case 2: if $g_k^T \ g_{k-1} \leq 0$, then $\beta_k^{ILB} \geq \beta_k^{FR}$. According to the β_k^{HS} β_k^{LS} formula then we have$\beta_k^{LS} \leq \beta_k^{ILB}$,$\beta_k^{LS} \leq \beta_k^{ILB}$ dividing both sides of 3.8 by $\left(g_{k-1}^T\ d_{k-1}\right)^2$, we get that

$$\frac{\|d_k\|^2}{\|g_{k-1}\|^2} = \frac{\|d_k\|^2}{\left(g_{k-1}^T d_{k-1}\right)^2} = \left(\beta_{k-1}^{ILB}\right)^2 \frac{\|d_{k-1}\|^2}{\left(g_{k-1}^T d_{k-1}\right)^2} - \frac{2h_{k-1}}{g_{k-1}^T} - \frac{h_{k-1}^2\|g_{k-1}\|^2}{\left(g_{k-1}^T\ d_{k-1}\right)^2}. \quad (3.11)$$

$$\leq \left(\beta_{k-1}^{LS}\right)^2 \frac{\|d_{k-1}\|^2}{\left(g_{k-1}^T d_{k-1}\right)^2} - \frac{2h_{k-1}}{g_{k-1}^T} - \frac{h_{k-1}^2\|g_{k-1}\|^2}{\left(g_{k-1}^T\ d_{k-1}\right)^2}, \quad (3.12)$$

$$\leq \left(\beta_{k-1}^{FR}\right)^2 + \frac{3\|g_{k-1}\|^4}{\|g_{k-1}\|^2} \frac{\|d_{k-1}\|^2}{\left(g_{k-1}^T d_{k-1}\right)^2} - \frac{2h_{k-1}}{g_{k-1}^T} - \frac{h_{k-1}^2\|g_{k-1}\|^2}{\left(g_{k-1}^T\ d_{k-1}\right)^2}, \quad (3.13)$$

$$= \left(\beta_{k-1}^{FR}\right)^2 \frac{\|d_{k-1}\|^2}{\left(g_{k-1}^T d_{k-1}\right)^2} + \frac{3\|d_{k-1}\|^2}{\|g_{k-1}\|^4} - \frac{(h_{k-1}-1)^2}{\|g_{k-1}\|^2} + \frac{1}{\|g_{k-1}\|^2}, \quad (3.14)$$

$$\leq \left(\beta_{k-1}^{FR}\right)^2 \frac{\|d_{k-1}\|^2}{\left(g_{k-1}^T d_{k-1}\right)^2} + \frac{3\|d_{k-1}\|^2}{\|g_{k-1}\|^4} + \frac{1}{\|g_{k-1}\|^2},$$

$$\leq 4 \sum_{k=1}^{n-1} \frac{1}{\|g_{k-1}\|^2} \leq 4\frac{K}{\varepsilon^3}. \quad (3.15)$$

If we assume $\| g_k \| \geq \varepsilon > 0$ for all k, then inequality (3.15) yields a uniform upper bound on a series involving $\frac{\|d_{k-1}\|^2}{\left(g_{k-1}^T d_{k-1}\right)^2}$. Using the descent property of d_{k-1} and the existence of a constant $\delta > 0$, with$-(g_{k-1}^t d_{k-1}) \geq \delta \parallel g_{k-1\parallel}{}^2$, we obtain a contradiction because the accumulated decrease of f would be infinite while f is bounded below. Hence $\lim_{k\to\infty} inf \parallel g_k \parallel = 0$.

The last inequalities imply that

$$\sum_{k=1} \frac{\|g_k-1\|^2}{\|d_{k-1}\|^2} \geq \ \varepsilon^2 \sum_{k\geq 1} \frac{1}{u_k} = \infty. \quad (3.16)$$

Which contradicts (11) in (Zhang & Zhou, 2008). We will amity it.

In the following of this section, we will introduce the global convergence of the new hybrid conjugate gradient method.

Theorem: Consider the new β_k^{ILB} hybrid conjugate gradient 3.1 and step size $\alpha_k$is obtained by 2.1 (Armijo) then the search direction d_{k-1} satisfies

$$g_k^T d_k < 0\ ,\ \beta_k^{IBL} \leq \frac{g_k^T d_k}{g_{k-1}^T d_{k-1}} = \beta_k^{FR}. \quad (3.17)$$

Proof: from theorem 3.2.2 we can see that the first part of (3.17) is true now, we will consider the second part we will divide in to two cases as follows. According to the algorithm 3.2 if $\beta_k^{ILB} > \beta_k^{FR} > 0$ then used formula (1.3), (1.4), (3.1).

The conclusion is true. If $\beta_k^{ILB} \leq \ \beta_k^{FR}$ and the Armijo line search satisfies (2.1) then we have

$$\beta_k^{ILB} \leq \frac{\|g_k+1\|^2 - \frac{\|g_{k+1}\|}{\|g_{k+1}\|} g_{k+1}^T\ g_{k+1}}{g_k^T d_k - g_{k-1}^T d_{k+1}} \leq \frac{\|g_{k+1}\|^2}{g_{k+1}^T d_k - g_k^T d_k} \leq \ \beta_k^{FR}.$$

By the use of lemma 3.1 and the same argument of the theorem 3.3 in (Dai & Yuan, 2001), we have the following global convergence result for the new hybrid conjugate gradient ethos.

Table 1. Comparison between the HSCG algorithm 2.1 and LSCG algorithm 2.2 with new hybrid algorithm 3.1 using test function different value of $\mathbf{Q_k}$ and for different value of n dimension the tools for each test functions.

Test functions	n-dim	CG algorithm for HS		CG algorithm for LS		New algorithm with Q = 0.05		New algorithm with Q = 0.5		New algorithm with Q = 0.9	
		NOI	NOF	NOI	NOF	NOI	NOF	NOI	NOF	NOI	NOF
Generalized Powell	5	88	252	120	243	48	54	50	56	50	56
	10	89	254	122	246	48	54	50	56	50	56
	100	104	308	127	301	48	54	50	56	50	56
	1000	120	243	129	323	48	54	50	56	50	56
Full Hessian	5	28	78	26	64	14	18	14	18	14	18
	10	32	160	29	67	14	18	14	18	14	18
	100	18	74	32	84	16	22	16	22	16	22
	1000	21	86	31	83	16	22	16	22	16	22
Diagonal-8	5	6	12	6	12	3	6	4	7	4	8
	10	6	12	6	12	3	6	4	7	4	8
	100	6	12	6	13	3	6	4	7	4	8
	1000	6	12	6	13	3	6	4	7	4	8
Non diagonal	5	44	188	126	372	24	60	26	63	26	63
	10	74	224	159	401	29	93	28	83	28	83
	100	129	302	286	667	49	121	44	138	46	140
	1000	407	602	403	899	47	140	46	142	47	142
Extended Deschnb	5	12	28	11	26	11	18	11	18	12	20
	10	13	29	13	31	11	18	11	18	12	20
	100	14	31	16	34	11	18	12	20	12	20
	1000	14	31	16	34	11	18	12	20	12	20
Generalized Tridiagonal	5	59	146	63	156	24	41	16	39	17	29
	10	59	148	63	156	27	43	17	29	17	29
	100	59	148	68	172	27	43	17	29	17	29
	1000	59	148	68	172	29	49	17	29	17	30
Generalized Dixon	5	128	264	115	219	112	204	115	219	115	219
	10	126	259	116	220	112	204	115	219	116	220
	100	126	259	116	220	118	214	116	220	116	220
	1000	126	259	116	220	118	214	116	220	116	220
eneralized Cantral	5	87	165	110	143	49	89	49	89	50	88
	10	87	166	121	156	49	89	49	89	50	88
	100	92	215	123	159	70	219	68	212	68	212
	1000	92	215	127	151	70	219	68	212	68	212
Generalized PSCl	5	58	118	43	89	14	23	16	28	18	30
	10	58	118	43	89	14	23	16	28	18	30
	100	59	120	43	89	14	23	16	31	18	30
	1000	59	120	43	89	14	23	16	31	18	30
Shanno	5	102	200	186	243	44	82	44	82	48	86
	10	104	206	189	254	46	82	46	83	48	86
	100	292	586	293	757	46	82	46	83	49	91
	1000	946	1763	193	1157	48	83	49	91	49	91
Generalized Cubic	5	10	40	20	62	5	16	6	17	8	18
	10	6	21	16	43	10	40	11	43	12	44
	100	13	48	16	46	20	60	11	43	12	44
	1000	21	74	24	83	20	60	31	122	31	46
Diagonal-7	5	6	20	8	28	3	15	3	16	4	18
	10	6	20	8	28	3	15	3	16	4	18
	100	6	20	8	28	3	15	3	16	4	18
	1000	6	20	8	28	3	15	3	16	4	18

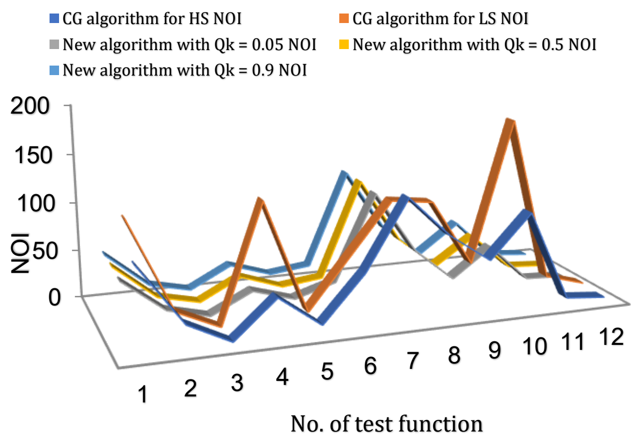

Figure 1. Comparison between the cg algorithm for HS (2.1) and LS (2.2) and cg algorithm with new algorithm (β_k^{ILB}) using different value of scalar Q_k=0.05, 0.5, 0.9 using NOI number of iteration function of dim 5 for each test well-known functions.

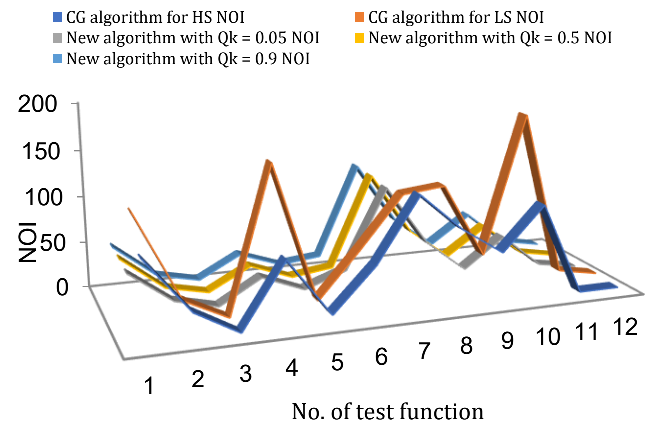

Figure 3. Comparison between the cg algorithm for HS (2.1) and LS (2.2) and cg algorithm with new algorithm (β_k^{ILB}) using different value of scalar Q_k=0.05, 0.5, 0.9 using NOI number of iteration function of dim 10 for each test well-known functions.

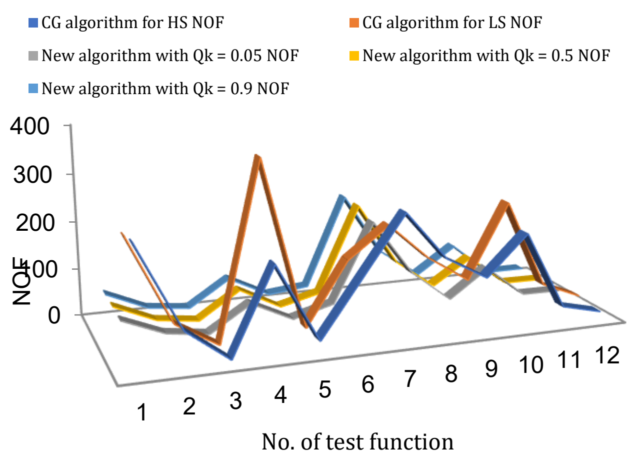

Figure 2. Comparison between the cg algorithm for HS (2.1) and LS (2.2) and cg algorithm with new algorithm (β_k^{ILB}) using different value of scalar Q_k=0.05, 0.5, 0.9 using NOF number of function evaluation of dim 5 for each test well-known functions.

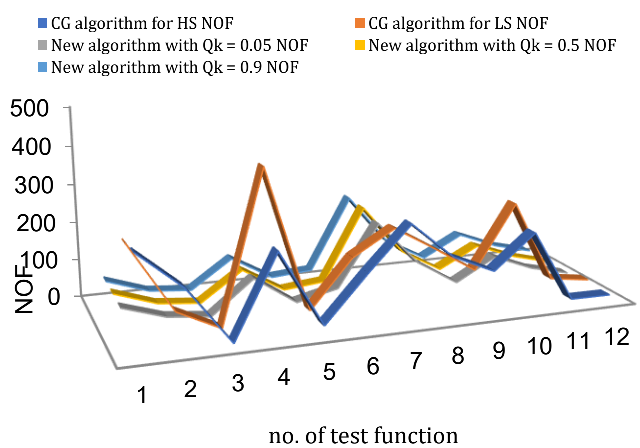

Figure 4. Comparison between the cg algorithm for HS (2.1) and LS (2.2) and cg algorithm with new algorithm (β_k^{ILB}) using different value of scalar Q_k=0.05, 0.5, 0.9 using NOF number of function evaluation of dim 10 for each test well-known functions.

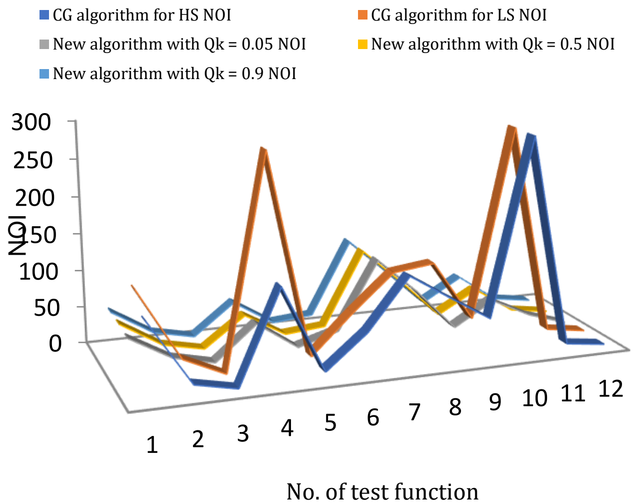

Figure 5. Comparison between the cg algorithm for HS (2.1) and LS (2.2) and cg algorithm with new algorithm (β_k^{ILB}) using different value of scalar Q_k=0.05, 0.5, 0.9 using NOI number of iteration function of dim 100 for each test well-known functions.

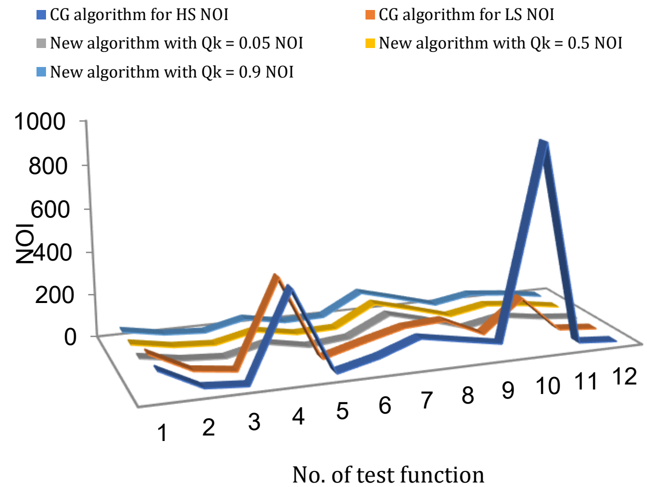

Figure 7. Comparison between the cg algorithm for HS (2.1) and LS (2.2) and cg algorithm with new algorithm (β_k^{ILB}) using different value of scalar Q_k=0.05, 0.5, 0.9 using NOI number of iteration function of dim 1000 for each test well-known function.

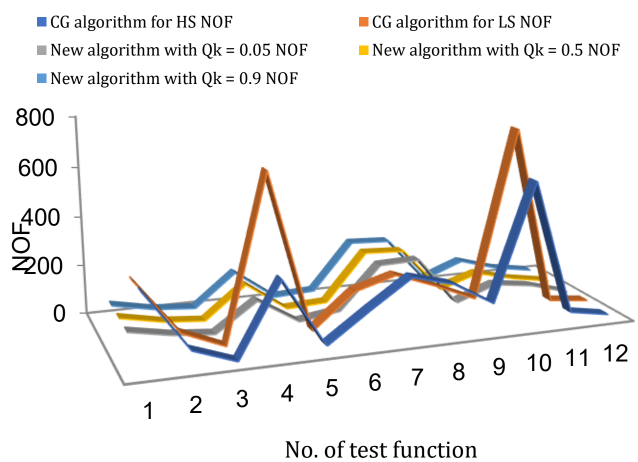

Figure 6. Comparison between the cg algorithm for HS (2.1) and LS (2.2) and cg algorithm with new algorithm (β_k^{ILB}) using different value of scalar Q_k=0.05, 0.5, 0.9 using NOF number of iteration function of dim 100 for each test well-known functions.

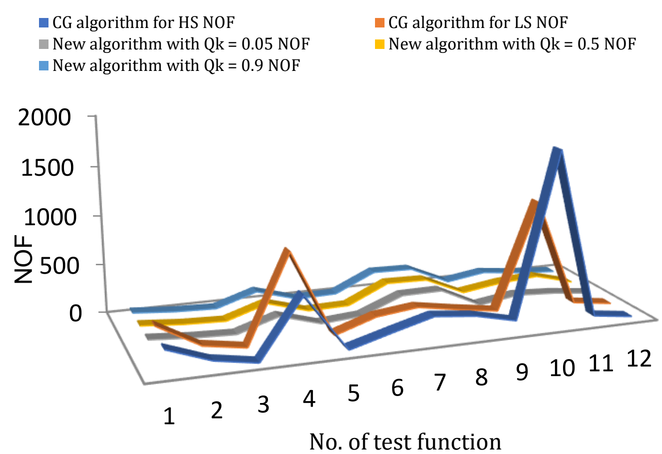

Figure 8. Comparison between the cg algorithm for HS (2.1) and LS (2.2) and cg algorithm with new algorithm (β_k^{ILB}) using different value of scalar Q_k=0.05, 0.5, 0.9 using NOF number of iteration function of dim 1000 for each test well-known functions.

8. Results

In this section, we evaluate the performance of the proposedhybrid conjugate gradient (β_k^{ILB}) method in a comparison with classical approaches such as Hestenes-Stiefel conjugate gradient method (HSCG), Liu-Storey conjugate gradient method

(LSCG). A wide range of standard test functions was employed, including quadratic, non-quadratic, unimodal, and multi-modal case with various dimensions (n=5, 10, 100, 1000) of unconstrained optimization problems. These problems are sourced from (Nocedal & Wright, 2006; Andrei, 2008; Mohammed et al., 2020). The experiments were conducted using (MATLAB 7.14), with the stopping criterion $\|g_k\| \leq 1 \times 10^{-5}$. For each problem, the starting point was randomly chosen to test the robustness of the methods under different initial conditions. The function utilized in this study is an artificial function. Artificial functions are selected to observe the algorithm's behaviour in various scenarios, such as narrow valleys, Unimodal functions, and functions with numerous significant local optima (Andrei, 2009). This paper tests ten nonlinear functions, as detailed in Table 1. To evaluate the performance profile of our method, we randomly selected the initial point for each test (Andrei, 2009).

The numerical results, summarized in Table 1 and illustrated in figures 1-8 demonstrate that the proposed method β_k^{ILB} achieves superior performance in terms of the number of iterations (NoI) in the new method β_k^{ILB} consistently requires fewer iterations compared to Hestenes-Stiefel method (HS) and Liu-Storey method (LS) across different test functions. Notably when $Q_k = 0.5$, the proposed algorithm achieves an average reduction of 30-40% in iteration relative to HS. Number function evaluations (NOF) are a critical measure of computational cost. Our method reduces NOF significantly, particularly in large-scale problems (n=1000) when HS and LS suffer from slow convergence. The proposed algorithm shows robustness across dimensions; unlike classical methods, which often degrade in performance as the problem dimension increases, it maintains stable efficiency. The scalar parameter Q_k plays a central role in the proposed algorithm by controlling the balance between HS and LS type updates, for small Q_k values, $Q_k = 0.05$ the algorithm tends to favours LS like behaviour, which is beneficial in early iterations but may lead to instability in certain cases. F or intermediate values $Q_k = 0.9$ the method behaves closer to HS producing more stable but slightly slower convergence. This adaptive mechanism explains why the proposed method consistently out performs both HS and LS, since it inherits their strengths while mitigating their weaknesses.

Overall, the results confirm that the proposed method is an effective large-scale hybrid CG method, highlighting its potential as a reliable tool for solving large-scale unconstrained optimization problems

9. Discussion

The proposed hybrid conjugate gradient method with an interchange scalar demonstrates both theoretical soundness and practical efficiency. Compared with classical HS and LS methods, the new approach shows faster convergence and reduced function evaluations, particularly in large-scale and high-dimensional optimization problems. These results are consistent with previous studies emphasizing the role of hybridization in improving robustness and efficiency (Andrei, 2009; Hager & Zhang, 2013). Moreover, the adaptive nature of the interchange scalar allows the method to balance between HS-like and LS-like updates, which stabilizes early iterations while ensuring long-term convergence. Similar adaptive strategies have been reported to enhance global convergence in unconstrained optimization (Dai & Yuan, 2001; Ibrahim & Mohammed, 2022).

Overall, the findings indicate that the proposed method is not only mathematically rigorous but also a reliable tool for solving complex nonlinear optimization problems across scientific and engineering applications.

10. Conclusion

We have introduced a novel conjugate gradient method that incorporates an interchange scalar for unconstrained nonlinear optimization. It unifies the advantages of HS and LS methods while avoiding their weaknesses. The algorithm has been thoroughly evaluated, demonstrating its effectiveness in achieving global solutions.

Extensive testing on standard optimization problems, including non-convex unconstrained problems, confirmed the global convergence of the new interchange scalar. Comparisons with existing methods indicate that the proposed hybrid method is both theoretically rigorous and practically efficient for large-scale optimization problems in science and engineering.

Conflict of Interest.

The authors declare that they have no conflict of interest related to the publication of this paper.

CRediT authorship contribution statement.

Acknowledgments.

The authors would like to express their sincere gratitude to Salahaddin University-Erbil for providing the academic environment and institutional support that made this research possible. We also extend our appreciation to our colleagues in the Departments of Mathematics and Computer Science for their constructive feedback and continuous encouragement throughout the preparation of this work.

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