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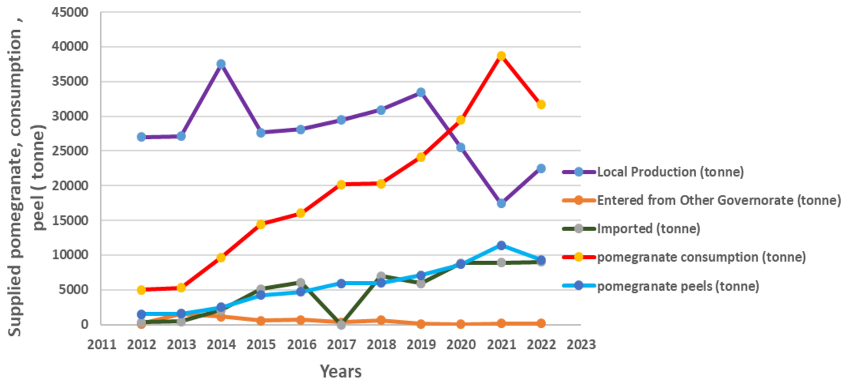

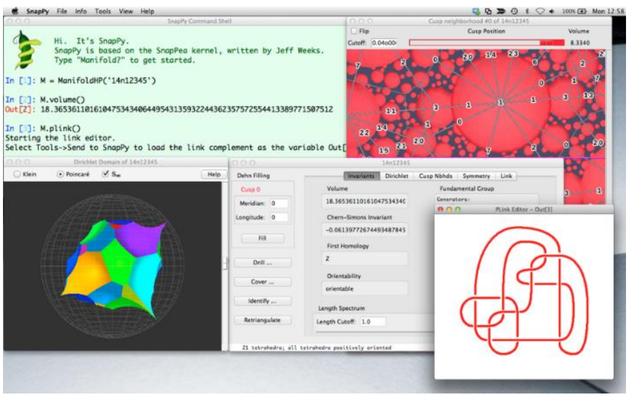

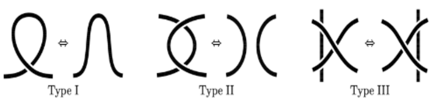

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A Minor Calculation On Co-Prime Knots And Links For p, q ≤ 20

Hero Mawlood Salih [1] , Hardi Ali Shareef [2] & Hardi Nasradeen Aziz [3]

[1] Departament of Mathematics, College of Education, University of Sulaimani, Sulaimaniah, Kurdistan Region, Iraq.
[2] Departament of Mathematics, College of Science, University of Sulaimani, Sulaimaniah, Kurdistan Region, Iraq.
[3] Departament of Mathematics, College of Education, University of Sulaimani, Sulaimaniah, Kurdistan Region, Iraq
*Corresponding authors' email: Hero.mawlood@gmail.com.

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Abstract

A knot K is a prime knot if it is not the unknot, in this paper, we present a novel classification of coprime knots and links, leveraging the calculus of continued fractions. Additionally, we provide an illustrative depiction of these coprime structures utilizing the SnapPy program, thereby obtaining crucial insights into their properties. The last section of this article presents a detailed table showing various knots and links, and their respective volumes, Schubert presentations with other important informations about them. So our investigation focuses on the coprime digits of knots, specifically those knots and links that are associated with the set of prime numbers from 1 to 20.

1.Introduction

The theory of knots stands as one of the oldest and most sophisticated branches of three-dimensional topology. At its heart lies the inquiry into how one topological space can be nested within another, which reveals its intricacies when considering seemingly simple objects like circles in R^3 or S^3. This exploration gives rise to the captivating and intricate domain known as knot theory. A knot can be envisioned as a closed loop of rope, where its ends are seamlessly joined. However, within the realm of mathematical abstraction, knots transcend their physical manifestations, becoming objects of study that encapsulate profound mathematical beauty and complexity. In this article, we narrow our focus to 3-dimensional cone manifolds, distinguished by a singular set taking the form of either a knot or a link. Our objective aligns with the broader pursuit of knot theory: to meticulously examine the myriad configurations in which a one-dimensional" string" can be arranged within a regular three-dimensional space. Two objects are homeomorphic if they can be deformed into each other by a continuous, invertible mapping.

The structure of this article presented as follows: Section two offers a concise overview of fundamental definitions pertinent to the theorem and our exploration, delving into the intricacies of knots and the point-wise behavior of prime digits in the continued fraction expansion. Section three navigates through the realm of continued fractions, employing them in the snappy to calculate coprime numbers. Moving forward, Section Four introduces the utilization of the SnapPy program for drawing knots and elucidates methods for extracting valuable insights into the behavior of these knots and links. Section Five presents a detailed tableau showcasing various knots and links, accompanied by their respective volumes, Schubert presentations, and additional pertinent information, offering a comprehensive glimpse into the diverse landscape of knot theory.[1]

2. Some Definitions and Notations

Definition 2.1. [8] A knot K is a subset of 3-dimensional Euclidean space R^3 that is homeomorphic to a circle.

Definition 2.2. [8] L is a link if L is a union of nonintersecting knots.
Definition 2.3. (Reidemeister moves) [6] Two diagrams represent equivalent knots or links if and only if the diagram two can be obtained from diagram one by a series of Reidemeister moves. Knowing that every knot and links admits a diagram.

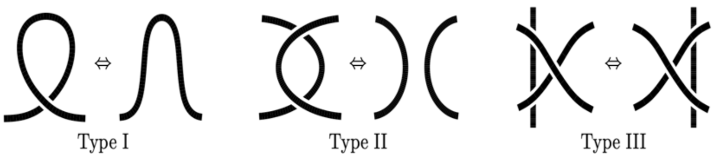

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Figure 1. Reidemeister moves.

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Definition 2.4. [10] Two knots are equivalent if there is continuous deformation (ambient isotopy) of S^3 taking one to the other.

Note: Every rational tangle $T\,(a_1\,,\dots,a_n\,)$ corresponds to a rational number of the form $\frac{p}{q}$ denoted by $[a_1, a_2, \dots\,, a_n]$.

Theorem 2.1. (Schubert, 1956) [5] Assume that rational tangles with functions $\frac{p}{q}$ and $\frac{p'}{q'}$ are given (p and q are relatively prime. Similarly for p' and q'). If K($\frac{p}{q}$) and K($\frac{p'}{q'}$) denote the corresponding rational knots acquired by taking numerator clousure of these tangles, then K($\frac{p}{q}$) and K($\frac{p'}{q'}$) are istopic if and only if

(1) $p \cong p'$ and

[1] This study is a collaborative effort with Keyvan Salehi and is aligned with the recent significant work by Gaven, Keyvan, and Yamashita [4].

(2) either $q \simeq q'$(mod p) or $qq' \simeq 1(mod\ p)$.

Theorem 2.2. (Conway's Theorem for Rational Links). [7] Let K_1 and K_2 be rational knots or links. Then K_1 and K_2 are equivalent if their continued fractions are equal.

Definition 2.5. Equivalent links[10]
Two links L_1 and L_2 are said to be equivalent if there is an orientation preserving picewise linear homeomorphism $h : S3 \rightarrow S3$ such that $h(L_1) = L_2$.

3. Co-Volume
Borel's machinery is a powerful method for finding lower bounds for co-volume. It was first introduced by Borel (see[2]). Let us mention fixing the following notations.

1. (1) N_ν : the norm of the prime ideal pν associated with the valuation ν of k.
2. (2) d_k: the discriminant of k.
3. (3) h_k: the class number of k $(= I(k)/P(k)|$ where $I(k)$ (resp. $P(k)$) is the group of fractional (resp. principal) ideals of k).

(4) $\zeta k(s)$: the Dedekind Zeta function of k.

(5) Rf (resp. R_∞): the set of finite (resp. infinite) places of k where A is ramified.

(6) O_k^*: the group of units of k.

(7) O_{Rf}^* : the group of R_f -units of k.

(8) $O_{R_f,+}^*$: the group of elements of units of O_{Rf}^* which are positive at all real places.

(9) $P\ (k, R_\infty)$: the principal ideals with a positive generator at all real places.
(10) M_1: the subgroup of $I(k)$ generated by $P(k, R_\infty)$ and the ideals p_ν such that $\nu \in R_f$.

(11) J_2: image of $P(k)$ in J_1.
(12) J_1: $I(k)/M_1$.
(13) $2J_1$: kernel of the mapping $y \rightarrow y^2$ in J_1.
(14) e: number of places of k dividing two and not contained in R_f .
(15) ϑ: Maximal order for ring A.

The proofs of the following theorems are given in ([2] -Theorem 7.3).

Theorem 3.1. Let Γ be a subgroup of $PGL2(C)$ commensurable with Γ_ϑ. Then the co-volume of H^3/Γ is an integral multiple of $2^{-e}V\ ol(H^3/\Gamma_\vartheta)$. It is equal to $V\ ol(H^3/\Gamma_\vartheta)$ if Γ is a maximal arithmetic group and $V\ ol(H^3/\Gamma_\vartheta)$ otherwise.

Theorem 3.2. Let ϑ be a maximal order of ring A. Then

$$Vol(H^3/\Gamma_\vartheta) = \frac{|d_k|^{\frac{2}{3}} \ \zeta_k(2)}{(4\pi^2)[k:Q]-1} \prod_{\nu \in R_f}(N\nu - 1) \times \frac{1}{\left[O_{R_f,+}^*: \left(O_{R_f}^* \ \right]\right)^2 [2J_1:J_2]} \qquad (3.1)$$

Here we calculate an example for finding the lower bound for $\mu(\Gamma)$, which is mentioned in tables. In other cases, calculations are the same.

4. Continued fraction and using it for calculating coprime numbers from 1 to 20

Continued fraction were introduced into knot theory by john H. Conway of ‘’ game of life” fame in his 1970 paper, ‘’ An enumeration of knots and links, and some of their algebraic properties.”[9]

Definition 4.1. [9] Co-prime numbers are those numbers that have only one common factor, namely 1.

For example 15 and 8 are coprime because 1 is the only common factors of 15 and 8, factors of 15 are 1, 3, 5, 15 and factors of 8 are 1, 2, 4, 8. More over 9 and 12 are not co-prime because they have two common factors; they are 1 and 3. HCF is 3 and they are not co-prime.

Definition 4.2. [9] A continued fraction is a finite or infinite expression of the form

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cdots}}}}$$

with convergence under appropriate conditions. The most commonly used form is where $b_i = 1$ for all i; such is called a regular continued fraction.

Thus, every rational tangle $T\,(a_1,\cdots,a_n)$ corresponds to a rational number of the form pq denoted by $[a_1, a_2,\cdots,a_n]$, Every real number may be (more or less) uniquely expressed as a finite or infinite regular continued fraction for which all the ai are integers and $a_i > 0$ for $i > 0$. One widely recognized outcome of the Euclidean algorithm is that, every rational number $\frac{p}{q}$ has a finite continued fraction expansion and only rational numbers have a finite continued fraction expansion.

Definition 4.3. [1]
A finite presentation $(X_I, X_2,\cdots,X_r \mid r_I, r_2,\cdots,r_s)$ is referred to as Wirtinger presentation if each relation r_i is presented as $X_h^{-1} w X_k w^{-1}$ for some letters X_h, X_k and a word w in $X_l, X_2,\cdots,X_r$.

5. Using SnapPy for drawing knots and links

In this section, we will illustrate which software program is used for drawing knots and links

Almost any prime knot with a few crossings can be identified using the following process, which takes ten to fifteen minutes for a novice user. After downloading and installing the program from the official website. Upon opening the program, one can type

M = Manifold()

to start the link editor. Sketch the knot’s form. To begin with, we only need to sketch a closed polygonal curve that follows the shape of the knot, without worrying about crossings. For example, we want to draw the diagram below. To start over if you make a mistake, select ”Clear” from the Tools menu.

You can alter which strand is on top by using your mouse to click on the crossings once you have drawn the knot's design. Here is my version of the knot:

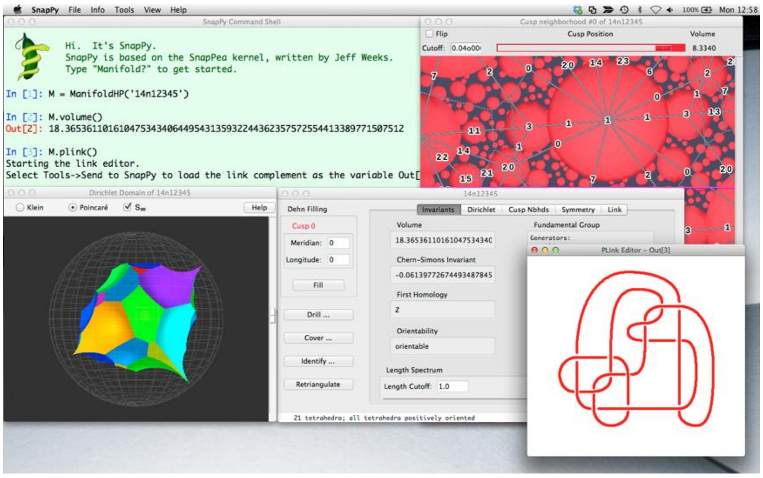


Figure 2. SnapPy program

Go to the "Tools" menu and select "Send to SnapPy". The SnapPy shell would look like this: then Type M.browse()
You will receive many descriptions of the manifold from the software, one of which will identify the co-prime knots and the volume, wirtinger presentation, as well as homology of those knots.

6. Classified co-prime knots and some information about them

Here we will find those links and knots which are isotopy, by using Lickorish Theorem.

Theorem 6.1. (Lickorish Theorem)[10] Suppose that for $p, q, \frac{p}{q} \cong \frac{p'}{q'}$ if and only if $p = p'$, $(qq' \cong_p 1, or\ q \cong_p q')$.

Remark 6.1. Using the definition of congruent modulo n, $n|(a - b)$ for integers a and b and $n \in N$, that is meaning $a \cong b(mod\ n)$, we have found that some of knots and links are equivalent.

Here are some examples:

$\frac{3}{5} \equiv \frac{3}{8}$ (mod 3)

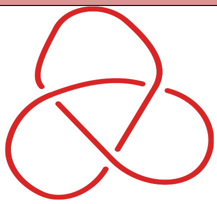
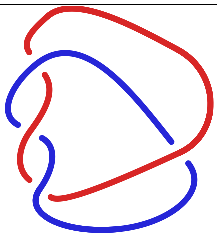
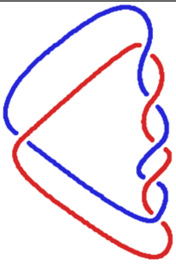
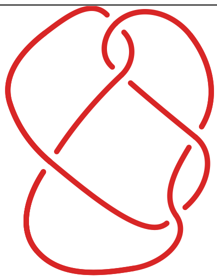
$\frac{3}{6} \equiv \frac{3}{9}(mod\ 3)$

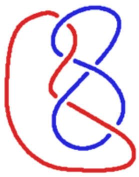
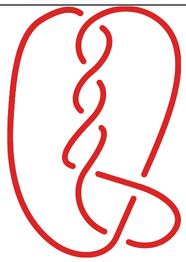
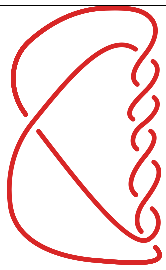
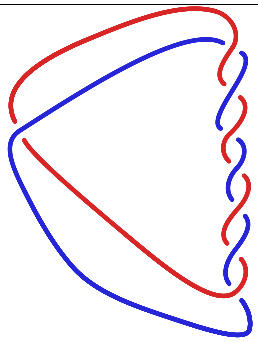
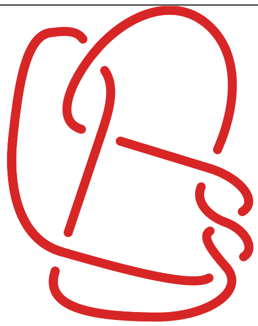
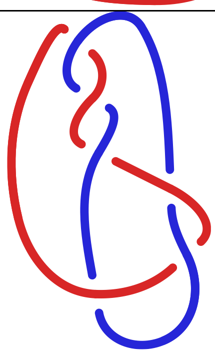
$\frac{4}{5} \equiv \frac{4}{9}(mod\ 4)$

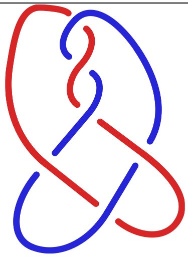
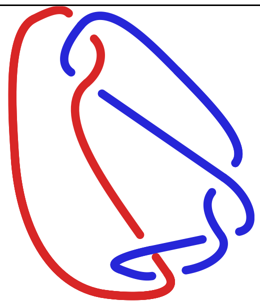
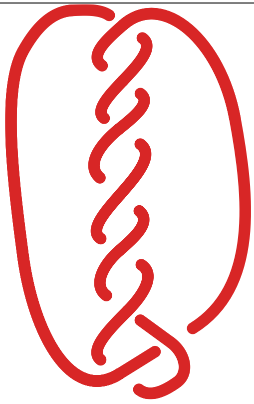
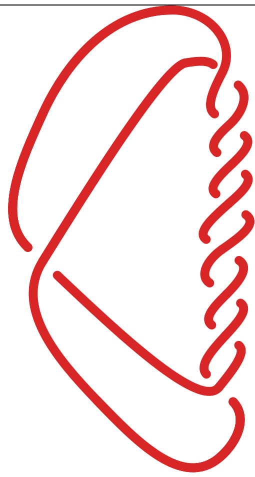
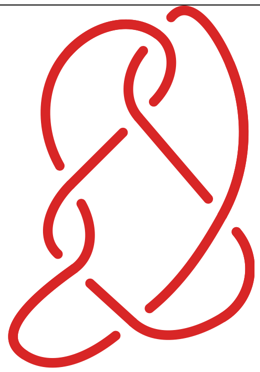
$\frac{5}{6} \equiv \frac{5}{11}(mod\ 5)$

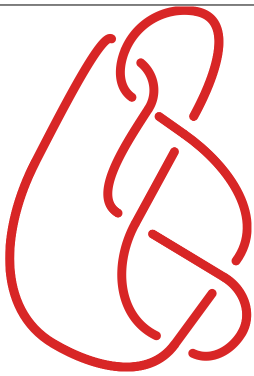
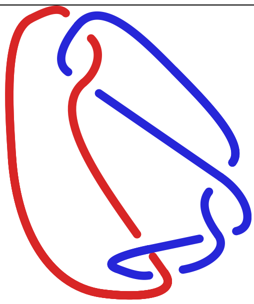
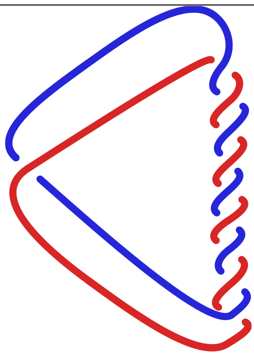
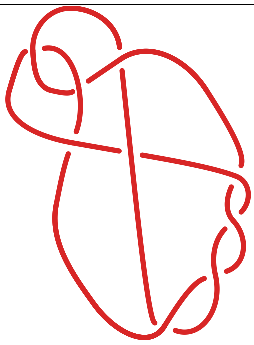
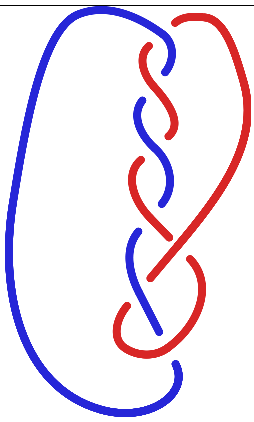
and so on.

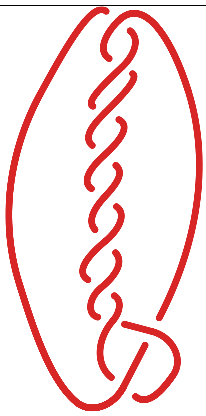
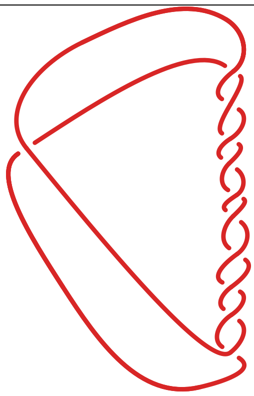
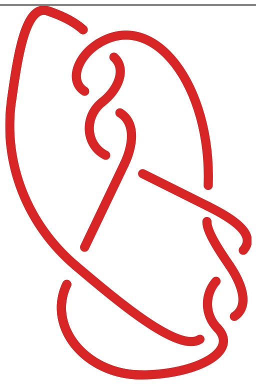
Table 1. The table below shows classified co-prime knots and links, and some information about them

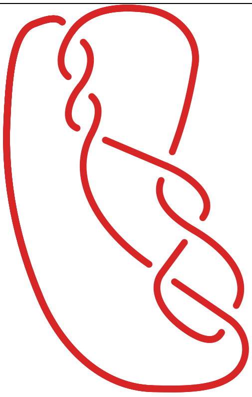
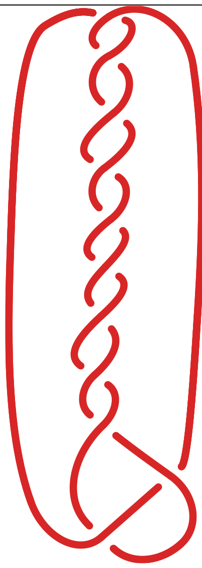
p/q	Conway Notation	Images	Volume	Homolog y	Wirtinger Presentation
$\frac{2}{3}$	[1,2]		2.2204460493 E-16	$\frac{Z}{2}$	a^2b^3
$\frac{3}{4}$	[1,3]		3.19623672363628 E- 5	$\frac{Z}{12}$	a^2b^4
$\frac{4}{5}$	[1,4]		1. E-5	$\frac{Z}{4}$	a^5b^2
$\frac{4}{7}$	[1,1,3]		2.7952004186	$\frac{Z}{4}$	a^5b^2
$\frac{5}{6}$	[1,5]		4.1633363423 E-15	$\frac{Z}{30}$	a^5b^6
$\frac{5}{6}$	[1,2,2]		0.0184831965483503	**Z+Z**	a^4b^7

$\frac{5}{8}$	[1,1,1,2]		2.64087989725	$\frac{Z}{40}$	a^8b^6
$\frac{5}{9}$	[1,1,4]		3.12584929611	$\frac{Z}{5}$	a^8b^9
$\frac{6}{7}$	[1,6]		3.E-5	$\frac{Z}{6}$	a^7b^2
$\frac{7}{8}$	[1,7]		5.4 E-8	$\frac{Z}{56}$	a^2b^8
$\frac{7}{9}$	[1,3,2]		3.1405549881	$\frac{Z}{7}$	a^4b^9
$\frac{7}{10}$	[1,2,3]		3.54486510332	$\frac{Z}{70}$	a^6b^{10}

$\frac{7}{11}$	[1,1,1,3]		4.3950704431	$\frac{Z}{7}$	$a^{10}b^{6}$
$\frac{7}{12}$	[1,1,2,2]		4.6471092705	$\frac{Z}{84}$	$a^{10}b^{12}$
$\frac{7}{13}$	[1,1,6]		3.40861523724	$\frac{Z}{7}$	$a^{12}b^{13}$
$\frac{8}{9}$	[1,8]		1.3703321537 E-14	$\frac{Z}{8}$	$a^{2}b^{9}$
$\frac{8}{11}$	[1,2,1,2]		4.3966382842	$\frac{Z}{8}$	$a^{9}b^{4}$

$\frac{\mathbf{8}}{\mathbf{13}}$	[1,1,1,1,2]		5.6825601628	$\frac{\boldsymbol{Z}}{\mathbf{8}}$	$a^{13}b^{10}$
$\frac{\mathbf{8}}{\mathbf{15}}$	[1,1,2,2]		4.6471092705	$\frac{\boldsymbol{Z}}{\mathbf{84}}$	$a^{10}b^{12}$
$\frac{\mathbf{9}}{\mathbf{10}}$	[1,9]		1.140754158E-14	$\frac{\boldsymbol{Z}}{\mathbf{9}}$	$a^{2}b^{10}$
$\frac{\mathbf{9}}{\mathbf{11}}$	[1,4,2]		3.14854227340	$\frac{\boldsymbol{Z}}{\mathbf{9}}$	$a^{9}b^{4}$
$\frac{\mathbf{9}}{\mathbf{14}}$	[1,1,1,4]		4.47628748706	$\frac{\boldsymbol{Z}}{\mathbf{26}}$	$a^{10}b^{14}$

$\frac{\mathbf{9}}{\mathbf{16}}$	[1,1,3,2]		5.73307310339	$\frac{\boldsymbol{Z}}{\mathbf{144}}$	$a^{16}b^{17}$
$\frac{\mathbf{9}}{\mathbf{17}}$	[1,1,8]		3.51526978116	$\frac{\boldsymbol{Z}}{\mathbf{9}}$	$a^{16}b^{17}$
$\frac{\mathbf{10}}{\mathbf{11}}$	[1,10]		3.71645579373553 E- 5	$\frac{\boldsymbol{Z}}{\mathbf{10}}$	$a^{2}b^{9}$
$\frac{\mathbf{10}}{\mathbf{13}}$	[1,3,3]		4.59021011389	$\frac{\boldsymbol{Z}}{\mathbf{10}}$	$a^{13}b^{12}$

$\mathbf{\frac{10}{17}}$	[1,1,2,3]		6.4418344385	$\frac{\boldsymbol{Z}}{\mathbf{10}}$	$a^{14}b^{17}$
$\mathbf{\frac{10}{19}}$	[1,1,9]		3.5485237487	$\frac{\boldsymbol{Z}}{\mathbf{10}}$	$a^{19}b^{4}$

Conclusion

In this article we sketched a new classification of coprime knots and links, using continued fractions, and we have illustrated a description of the co-prime knots and links using the snapPy program. We obtained vital information about them. The last section of this article has a comprehensive table that displays different knots and links connections together with the volumes that correspond to them. It also included Schubert presentations. In the present paper, we are interested in the co-prime digits of knots, i.e. those knots and links that happen to belong to the set P from 1 to 20 of prime numbers.

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